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On the Hyperelliptic Sigma Functions.

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Introduction.

It is known of what importance for the development of the theory of elliptic functions is the equation

$$\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2(u)\,\sigma^2(v)} = \frac{d^2}{du^2}\log\sigma u - \frac{d^2}{dv^2}\log\sigma v.$$

It becomes therefore a matter of interest to enquire whether an analogous equation holds for higher cases. Denoting a theta function with half-integer characteristic A by $\vartheta(u;\,A)$, and a theta function whose characteristic is the sum of the half-integer characteristics A, B, by $\vartheta(u;\,AB)$, it is known that there exist, in the general case of Abelian functions, $\frac{1}{2}p(p+1)$ equations of the form

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A) = \sum_r C_r\frac{\vartheta^2(u;\,AP_r)}{\vartheta^2(u;\,A)},$$

wherein A is an arbitrary characteristic, P_r is one of a group of 2^p characteristics over which the summation on the right side is to extend, C_r is a constant, and u_i, u_j are any two of the p arguments $u_1, \ldots\ldots, u_p$ (Frobenius, Crelle LXXXIX, 1880). These equations furnish $\frac{1}{2}p(p+1)$ conditions for the expression of the $2^p - 1$ quotients $\vartheta^2(u;\,AP_r)/\vartheta^2(u;\,A)$ in terms of functions of the form $\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A)$. If these equations were capable of solution, we might expect, by means of the addition equation of the theta functions, to be able to obtain the function $\vartheta(u+v;\,A)\,\vartheta(u-v;\,A)/\vartheta^2(u;\,A)\,\vartheta^2(v;\,A)$ in terms of functions $\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\,A)$. Except in case $p=2$, in which $\frac{1}{2}p(p+1) = 3 = 2^p - 1$, it is not however obvious* that the equations are capable of solution. It is therefore of interest, and it is the final (though not the only) purpose of this paper to prove, that *in the hyperelliptic case, for any value of p, the required expression can be effected.* To be more precise, it is shown below, that *if Q_i be any one of a certain group of 2^p characteristics, then, in the hyperelliptic case*

$$\frac{\vartheta(u+v;\ Q_i)\,\vartheta(u-v;\ Q_i)}{\vartheta^2(u;\ Q_i)\,\vartheta(v;\ Q_i)}$$

can be expressed as an integral polynomial in the $p\,(p+1)$ functions

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u;\ Q_i),\qquad \frac{\partial^2}{\partial v_i\,\partial v_j}\log\vartheta(v;\ Q_i).$$

I have desired to arrive at this result, starting only from the general results associated with Riemann's theory of Abelian functions, and without assuming properties for which, though in some cases their general character is already well known, I was not able to give a precise reference to the existing literature. In some cases, to save space, I have ventured to give references to my recent book on Abelian functions; these references are printed thus, [B. 25]. An analysis of the steps of the argument which leads to the final result, is prefixed to the development. The formulæ in Nos. VII and VIII of this analysis would also appear to be of importance in the theory of the hyperelliptic theta functions.

ANALYSIS.

I.—*Of the method of the paper.*

We suppose the fundamental algebraical equation to be

$$y^2 = 4P(x)\,Q(x),$$

wherein

$$Q(x) = (x-c_1)(x-c_2)\ldots\ldots(x-c_p)(x-c),$$
$$P(x) = (x-a_1)(x-a_2)\ldots\ldots(x-a_p),$$

that is, we suppose one of the branch places to be at infinity. We suppose the cross lines of the Riemann surface, by which we pass from one sheet to the other, to be $c_1a_1,\ c_2a_2,\ \ldots\ldots,\ c_pa_p,\ ca,$ where a denotes the branch place at infinity, and we call $c_1, a_1, c_2, a_2, \ldots . c_p, a_p, c, a$, the *ascending order* of the branch places; we do not however suppose the quantities c_1, a_1, etc., to be real The dissection of the surface, whereby it is changed into the p-ply connected surface on which the Abelian integrals are single valued, is taken to be that denoted by the annexed diagram. Other methods of dissection are discussed below; from the discussion the reasons for the adoption of this method will appear.

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*Apart from the general theorems announced by Weierstrass, Crelle, LXXXIX, 1880, p. 7, from which such a result was to be expected.

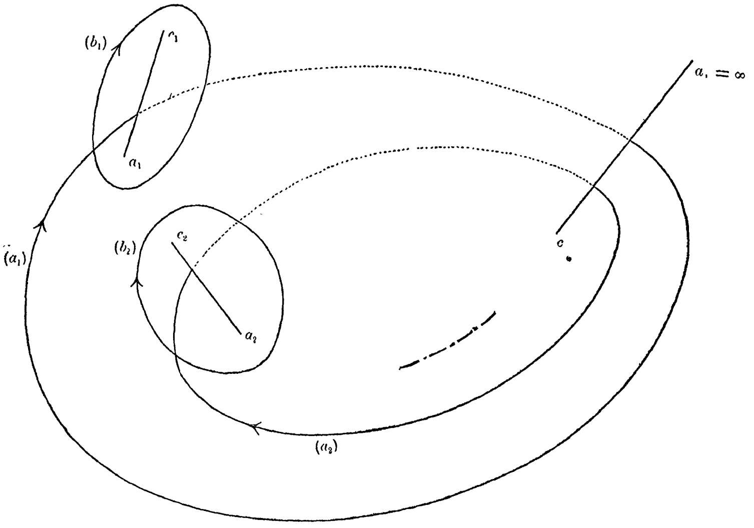

FIG. A.

[The period of any Abelian integral at a period loop is the constant by which the value of the integral on the side of the loop which is on the left when the loop is described in the direction of the arrow-head exceeds the value of the integral on the other side. The loops themselves are called by the letters (b_1), (a_1), ... here placed in brackets. The loops (a_1), (a_2), ... are called the first, second, ... period loops of the first kind, the loops (b_1), (b_2), ... are called the first, second, ... period loops of the second kind. In the figure $p - 2$ pairs of loops are not shown.]

The finite branch places, taken in the ascending order, are frequently denoted, respectively, by $b_1, b_2, b_3, b_4, \ldots . b_{2p-1}, b_{2p}, b_{2p+1}$. In some cases, however, where no misunderstanding can arise, a batch of k finite branch places, taken in the ascending order but not necessarily beginning with the branch place c_1, is denoted by $b_1, b_2, \ldots ., b_k$.

It is known that any one of the 2^{2p} theta functions arising from the fundamental algebraic equation

$$y^2 = (x,\ 1)_{2p+2} = f(x)$$

is associated with a decomposition of the integral polynomial $f(x)$ into two factors, in the form

$$f(x) = \phi_x^{p+1-2\mu}\,\psi_x^{p+1+2\mu};$$

in what follows we are primarily, though not exclusively, concerned with those 2^p functions which arise for all decompositions in which $\psi_x^{p+1+2\mu}$ has a fixed factor of order $p+1$, namely, the factor which we have denoted by $Q(x)$. Cf. below, No. VII of the Analysis.

II.—*Of two signs depending on the dissection of the Riemann surface.*

If

$$Q = \tfrac{1}{2}\binom{q'}{q} = \tfrac{1}{2}\begin{pmatrix} q_1',\ q_2',\ \ldots\ldots,\ q_p' \\ q_1,\ q_2,\ \ldots\ldots,\ q_p \end{pmatrix}, \quad K = \tfrac{1}{2}\binom{k'}{k} = \tfrac{1}{2}\begin{pmatrix} k_1',\ k_2',\ \ldots\ldots,\ k_p' \\ k_1,\ k_2,\ \ldots\ldots,\ k_p \end{pmatrix}$$

be two characteristics of half-integers, we use (after Frobenius) the abbreviations

$$|\,Q,\ K\,| = \sum_{r=1}^{p}(q_r k_r' - q_r' k_r), \quad \binom{Q}{K} = e^{\pi i \sum_{r=1}^{p} q_r' k_r}.$$

Further, b denoting any one of the $2p+1$ finite branch places, and $2\omega_{r,\,i}$, $2\omega'_{r,\,i}$ denoting the periods of an integral of the first kind, $u_r^{x,\,a}$, at the i-th period loops respectively of the first and second kind, if

$$u_r^{a,\,b} = \beta_1\omega_{r,\,1} + \ldots\ldots + \beta_p\omega_{r,\,p} + \beta_1'\omega'_{r,\,1} + \ldots\ldots + \beta_p'\omega'_{r,\,p},$$

(so that, as is known, $\beta_1, \ldots\ldots, \beta_p'$ are integers), we put

$$B = \tfrac{1}{2}\binom{\beta'}{\beta}.$$

In regard then to the $2p+1$ half-integer characteristics, B, thus arising, it is important to consider the values of the two quantities

$$e^{\frac{1}{2}\pi i\,|\,B_i,\ B_j\,|}, \quad \binom{B_i}{B_j},$$

wherein B_i, B_j are any two of the $2p+1$ characteristics in question. The origin of these two quantities will be seen in the two following Nos. III, IV respectively. These signs may be replaced by the single one

$$\sqrt{\left(\begin{matrix} Q \\ K \end{matrix}\right)} = e^{\frac{1}{2}\pi i \sum_{r=1}^{p} q_r' k_r};$$

but this is practically not very convenient.

III.—*On the fundamental radical functions.*

On the p-ply connected dissected Riemann surface upon which the Abelian integrals are single valued, there exist $2p+1$ single valued functions, of which the squares are the $2p+1$ functions $x-b$, b denoting any one of the finite branch places. Supposing the values of y to be beforehand allocated to the places of the Riemann surface, they shall be defined by the facts, (i) that at infinity the ratio of any two of them is unity, (ii) that their product is equal to $\frac{1}{2}y$. We denote them by the symbols $\sqrt{x-b}$. We are then able to define the symbols $\sqrt{b_i - b_j}$ as the value of $\sqrt{x-b_j}$ at the place b_i. With this definition we are able to prove the equation

$$\sqrt{b_i - b_j} = \sqrt{b_j - b_i} \,.\, e^{\frac{1}{2}\pi i \,|B_i,\, B_j|},$$

which holds for any method of dissection of the Riemann surface.

IV.—*Of the expression of a certain theta-quotient.*

If $u_1^{x,\,a}, \ldots\ldots, u_p^{x,\,a}$ be a system of linearly independent integrals of the first kind, and, as usual, $a_1, \ldots\ldots, a_p$ being the branch places before described,

$$u_r = u_r^{x_1,\,a_1} + \ldots\ldots + u_r^{x_p,\,a_p}, \qquad (r = 1, 2, \ldots\ldots, p)$$

then the theta quotient

$$\frac{\vartheta^2(u\,|\,B_i B_j)\,\vartheta^2(u)}{\vartheta^2(u\,|\,B_i)\,\vartheta^2(u\,|\,B_j)}$$

is equal to
$$\left(\begin{matrix} B_i \\ B_j \end{matrix}\right)(b_i - b_j)\left\{\frac{1}{2}\sum_{r=1}^{p}\frac{y_r}{(x_r - b_i)(x_r - b_j)\,F'(x_r)}\right\}^2,$$

where
$$F(x) = (x - x_1)(x - x_2)\ldots\ldots(x - x_p).$$

(Cf., for instance, Bolza, American Journal, XVII (1895), where, however, the sign of the right side is not made precise.)

V.—*Of the dissection of the Riemann surface.*

It is possible to take the period loops, on the Riemann surface, so that, *when* $i > j$, the value of each of the quantities $|B_i, B_j|$, $\left(\begin{smallmatrix} B_i \\ B_j \end{smallmatrix}\right)$ shall be independent of i and j. This is manifestly a convenience. These two quantities are however independent; it is possible to choose dissections in which their values form any one of the four combinations $(+1, +1)$, $(-1, -1)$, $(+1, -1)$, $(-1, +1)$. We adopt as our standard method of dissection one in which *when* $i > j$,

$$|B_i, B_j| = -|B_j, B_i| = +1,$$

$$\begin{pmatrix} B_i \\ B_j \end{pmatrix} = -\begin{pmatrix} B_j \\ B_i \end{pmatrix} = +1.$$

Then we have, when $i > j$,

$$\sqrt{b_j - b_i} = -i\sqrt{b_i - b_j},$$

$$\frac{\vartheta^2(u\,|\,B_i B_j)\,\vartheta^2(u)}{\vartheta^2(u\,|\,B_i)\,\vartheta^2(u\,|\,B_j)} = (b_i - b_j)\left\{\tfrac{1}{2}\sum_{r=1}^{p}\frac{y_r}{(x_r - b_i)(x_r - b_j)\,F'(x_r)}\right\}^2.$$

VI.—*Resulting preliminary formulæ. Introduction of sigma-functions.*

Let

$$u_r^{x,\,a} = \int_a^x \frac{x^{r-1}dx}{y},$$

and let the value, when the arguments $u_1, \ldots\ldots, u_p$ are zero, of the expression

$$\frac{1}{\vartheta(0)}\,\frac{1}{i\sqrt{f'(a_r)/4}}\left(\frac{\partial}{\partial u_1} + a_r\frac{\partial}{\partial u_2} + \ldots\ldots + a_r^{p-1}\frac{\partial}{\partial u_p}\right)\vartheta(u\,|\,A_r),$$

wherein A_r is the half-integer characteristic associated with the half-period $u^{a,\,a_r}$, be denoted by λ_r; herein, as always in what follows, $i\sqrt{f'(a_r)/4}$ is written to denote

$$i\sqrt{a_r - c_1}\,\sqrt{a_r - a_1}\,\sqrt{a_r - c_2}\,\ldots\ldots\,\sqrt{a_r - c},$$

the symbols $\sqrt{a_r - c_1}$, etc., being as in No. III; then, for a proper determination of the sign of the denominator, we have

$$\lambda_r = \frac{(-1)^{p-r}\sqrt{P'(a_r)}}{(i\sqrt{f'(a_r)/4})^{\frac{1}{2}}} = \frac{\sqrt{(-1)^{p-r}P'(a_r)}}{(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4})^{\frac{1}{2}}},$$

where $\sqrt{P'(a_r)}$ denotes

$$\sqrt{a_r - a_1}\,\sqrt{a_r - a_2}\,\ldots\ldots\,\sqrt{a_r - a_p}\,,$$

the symbols $\sqrt{a_r - a_1}$, etc., being as in No. III, and $\sqrt{(-1)^{p-r}P'(a_r)}$ denotes

$$\sqrt{a_r - a_1}\,\ldots\ldots\,\sqrt{a_r - a_{r-1}}\,\sqrt{a_{r+1} - a_r}\,\ldots\ldots\,\sqrt{a_p - a_r}\,.$$

Further we have

$$\frac{\vartheta(u\,|\,A_r)}{\vartheta(u)} = i\,\frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4})^{\frac{1}{2}}} = \frac{\sqrt{F(a_r)}}{(i\sqrt{f'(a_r)/4})^{\frac{1}{2}}} = (-1)^{p-r}\lambda_r\,\frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}}\,,$$

where

$$\begin{aligned}\sqrt{(-1)^{p-r+1}F(a_r)} &= \sqrt{a_r - x_1}\,\ldots\ldots\,\sqrt{a_r - x_{r-1}}\,\sqrt{x_r - a_r}\,\ldots\ldots\,\sqrt{x_p - a_r}\,,\\ \sqrt{F(a_r)} &= \sqrt{a_r - x_1}\,\ldots\ldots\,\sqrt{a_r - x_p}\,;\end{aligned}$$

we have previously (No. III) defined the sense in which the symbols $\sqrt{x-b}$ are used; it is convenient also, as here, to introduce symbols denoted by $\sqrt{b-x}$; we *define* then the symbol $\sqrt{b-x}$, wherein b is any one of the $2p+1$ finite branch places, by the equation

$$\sqrt{x-b} = -i\sqrt{b-x}\,.$$

Such formulæ as these have of course been given before (e. g. Königsberger, Crelle, LXIV (1865), for the case when the branch values $c_1, a_1, c_2, \ldots\ldots$ are real); but, I think, without the precise determination of the signs of the square roots involved which is necessary for the deductions to be drawn in this paper.

Further, B_r, B_s, denoting as before the characteristics associated with the half-periods $u^{a,\,b_r}$, $u^{a,\,b_s}$, and B_rB_s denoting the sum of these characteristics, *without reduction*, we put

$$\frac{\vartheta(u\,|\,B_rB_s)\,\vartheta(u)}{\vartheta(u\,|\,B_r)\,\vartheta(u\,|\,B_s)} = \varepsilon_{r,\,s}\sqrt{b_r \sim b_s}\left\{\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i - b_r)(x_i - b_s)\,F'(x_i)}\right\},$$

where $b_r \sim b_s$ is written, instead of $b_r - b_s$, to denote that in the ascending order of the branch places, b_r has a higher place than b_s, and $\varepsilon_{r,\,s}$ is a square root of unity, in fact defined by this equation, which we do not determine.

Then we find, if $\zeta_{r,\,s} = \varepsilon_{r,\,s}\sqrt{b_r \sim b_s}$, a notation which will be frequently employed, that

$$\frac{\vartheta(u\,|\,B_rB_sB_t)\,\vartheta^2(u)}{\vartheta(u\,|\,B_r)\,\vartheta(u\,|\,B_s)\,\vartheta(u\,|\,B_t)} = -\,\zeta_{r,\,s}\zeta_{s,\,t}\,\zeta_{r,\,t}\left\{\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_r)(x_i-b_s)(x_i-b_t)\,F'(x_i)}\right\}.$$

It is to be noticed that $\zeta_{r,\,s} = \zeta_{s,\,r}$, etc., as follows from the definition.

We now suppose b_r, b_s, b_t to be chosen from the branch places $a_1, \ldots\ldots, a_p$; and we introduce the definitions

$$\begin{aligned}\sigma_r(u) &= \quad \lambda_r\,\vartheta(u\,|\,A_r)/\vartheta(0),\\ \sigma_{r,\,s}(u) &= -\,\frac{\lambda_r\lambda_s}{\zeta_{r,\,s}}\,\vartheta(u\,|\,A_rA_s)/\vartheta(0),\\ \sigma_{r,\,s,\,t}(u) &= +\,\frac{\lambda_r\lambda_s\lambda_t}{\zeta_{r,\,s}\,\zeta_{s,\,t}\,\zeta_{r,\,t}}\,\vartheta(u\,|\,A_rA_sA_t)/\vartheta(0),\end{aligned}$$

wherein A_r is the characteristic associated with the half-period $u^{a,\,a_r}$, and A_rA_s denotes the sum of the characteristics A_r, A_s, without reduction, etc. Then, bearing in mind that the argument u_k is given by

$$u_k = \int_{a_1}^{x_1}\frac{x^{k-1}dx}{y} + \ldots\ldots + \int_{a_p}^{x_p}\frac{x^{k-1}dx}{y}, \qquad (k = 1, \ldots\ldots, p)$$

we find that the first terms in the expansion of $\sigma_r(u)$ in a series of integral positive powers of $u_1, \ldots\ldots, u_p$ are the linear terms which may symbolically be denoted by

$$\begin{aligned}\frac{P(\xi)}{\xi - a_r}, &= (\xi - a_1)(\xi - a_2)\ldots\ldots(\xi - a_p),\\ &= \xi^{p-1} + A_1\xi^{p-2} + A_2\xi^{p-3} + \ldots\ldots + A_{p-1}, \text{ say,}\end{aligned}$$

provided we replace the powers of the symbolical quantity ξ according to the definitions

$$\xi^{i-1} = u_i, \quad \xi^0 = u_1; \qquad (i = 1, 2, \ldots\ldots, p)$$

that is, the linear terms are

$$u_p + A_1u_{p-1} + A_2u_{p-2} + \ldots\ldots + A_{p-1}u_1.$$

Similarly the first terms in the expansion of $\sigma_{r,\,s}(u)$ are linear terms given, symbolically, by

$$\begin{aligned}\frac{P(\xi)}{(\xi - a_r)(\xi - a_s)}, &= \xi^{p-2} + \xi^{p-3}B_1 + \xi^{p-4}B_2 + \ldots\ldots + B_{p-2}, \text{ say,}\\ &= u_{p-1} + B_1u_{p-2} + B_2u_{p-3} + \ldots\ldots + B_{p-2}u_1;\end{aligned}$$

and the first terms in the expansion of $\sigma_{r,\,s,\,t}(u)$ are quadratic terms given symbolically by

$$\tfrac{1}{2}\phi(\xi)\,\phi(\xi')(\xi-\xi')^2,$$

where

$$\phi(\xi)=\frac{P(\xi)}{(\xi-a_r)(\xi-a_s)(\xi-a_t)},$$

and, after the division has been carried out, we are to replace the equivalent symbols ξ, ξ' according to the laws

$$\xi^{i-1}=u_i=\xi'^{\,i-1},\quad \xi^0=u_1=\xi'^0.\qquad (i=1,\,2,\,\ldots\ldots,\,p)$$

The function $\sigma_r(u)$ is one of p functions; the function $\sigma_{r,\,s}(u)$ is one of $\frac{1}{2}p(p-1)$ functions; the function $\sigma_{r,\,s,\,t}(u)$ is similarly one of $\frac{1}{6}p(p-1)(p-2)$ functions.

These expansions show that the functions agree with the hyperelliptic sigma functions considered by Klein. Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442.

VII.—*New expression of theta functions of three or more suffixes in terms of functions of one or two suffixes.*

Let $b_1,\,b_2,\,\ldots\ldots,\,b_{2n+1}$ be any finite branch places, taken in any order; let

$$\phi_{r,\,s}(u)=\frac{\vartheta(u\,|\,B_rB_s)}{\zeta_{r,\,s}},$$

the notation being as before, and put

$$\nabla_{12\,\ldots\ldots\,k}=\zeta_{1,\,2}\,\zeta_{1,\,3}\,\zeta_{2,\,3}\,\zeta_{1,\,4}\,\zeta_{2,\,4}\,\zeta_{3,\,4}\ldots\ldots\zeta_{1,\,k}\,\zeta_{2,\,k}\ldots\ldots\zeta_{k-1,\,k};$$

let the suffixes 1, 2, ..., $2n$ be divided into two sets $r_1,\,\ldots\ldots,\,r_n$ and $s_1,\,\ldots,\,s_n$, and put

$$D_n=(b_{r_1}-b_{r_2})\ldots\ldots(b_{r_1}-b_{r_n})(b_{r_2}-b_{r_3})\ldots\ldots(b_{r_2}-b_{r_n})\ldots\ldots(b_{r_{n-1}}-b_{r_n}),$$
$$E_n=(b_{s_1}-b_{s_2})\ldots\ldots(b_{s_1}-b_{s_n})(b_{s_2}-b_{s_3})\ldots\ldots(b_{s_2}-b_{s_n})\ldots\ldots(b_{s_{n-1}}-b_{s_n});$$

then, if $\vartheta_{1,\,2\,\ldots\ldots\,k}(u)$ denote $\vartheta(u\,|\,B_1B_2\ldots\ldots B_k)$, where $B_1\ldots\ldots B_k$ denotes the sum, without reduction, of the half-integer characteristics associated with the half-periods $u^{a,\,b_1},\,\ldots\ldots,\,u^{a,\,b_k}$, we have the theorem

$$\vartheta^{n-1}(u)\,\vartheta_{12\,\ldots\ldots\,2n}(u)=\begin{vmatrix}\phi_{r_1,\,s_1}, & \phi_{r_1,\,s_2}, & \ldots\ldots, & \phi_{r_1,\,s_n}\\ \phi_{r_2,\,s_1}, & \phi_{r_2,\,s_2}, & \ldots\ldots, & \phi_{r_2,\,s_n}\\ \ldots & \ldots & \ldots & \ldots\\ \phi_{r_n,\,s_1}, & \phi_{r_n,\,s_2}, & \ldots\ldots, & \phi_{r_n,\,s_n}\end{vmatrix}\frac{\nabla_{12\,\ldots\ldots\,2n}}{D_nE_n}$$

Further, if the suffixes 1, 2, ... , $2n+1$ be divided into two sets $r_1, \dots, r_n$ and $s_1, \dots\dots, s_{n+1}$, and D_n be defined as before, but

$$E_{n+1} = E_n\,(b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}}) \dots\dots (b_{s_n} - b_{s_{n+1}}),$$

then we have the theorem

$$\vartheta_n(u)\,\vartheta_{12\dots\dots 2n+1}(u) = \begin{vmatrix} \vartheta_{s_1}, & \vartheta_{s_2}, & \dots\dots, & \vartheta_{s_{n+1}} \\ \phi_{r_1, s_1}, & \phi_{r_1, s_2}, & \dots\dots, & \phi_{r_1, s_{n+1}} \\ \dots & \dots & \dots & \dots \\ \phi_{r_n, s_1}, & \phi_{r_n, s_2}, & \dots\dots, & \phi_{r_n, s_{n+1}} \end{vmatrix} \frac{\nabla_{12\dots\dots 2n+1}}{D_n E_{n+1}}.$$

In these equations ϑ_{s_1} is written for $\vartheta(u\,|\,B_{s_1})$, and $\phi_{r,s}$ for $\phi_{r,s}(u)$.

We now suppose the branch places $b_1, \dots\dots, b_{2n+1}$ to be chosen from among the places $a_1, \dots\dots, a_p$; and we put

$$\begin{aligned} \sigma(u) &= \frac{\vartheta(u)}{\vartheta(0)}, \\ \sigma_{12\dots\dots 2n}(u) &= (-1)^n \frac{\lambda_1\lambda_2 \dots\dots \lambda_{2n}}{\nabla_{12\dots\dots 2n}} \frac{\vartheta(u\,|\,A_1 \dots\dots A_{2n})}{\vartheta(0)}, \\ \sigma_{12\dots 2n+1}(u) &= \frac{\lambda_1\lambda_2 \dots\dots \lambda_{2n+1}}{\nabla_{12\dots\dots 2n+1}} \frac{\vartheta(u\,|\,A_1 \dots\dots A_{2n+1})}{\vartheta(0)}, \end{aligned}$$

wherein $A_1 \dots\dots A_{2n}$ denotes the sum of the characteristics associated with the half-periods $u^{a, a_1}, \dots\dots, u^{a, a_{2n}}$, and $2n+1$ is supposed not greater than p, so that we define, hereby, 2^p sigma functions. Then we find

$$\sigma^{(n-1)}(u)\,.\,\sigma_{12\dots\dots 2n}(u) = \begin{vmatrix} \sigma_{r_1, s_1}(u), & \dots\dots, & \sigma_{r_1, s_n}(u) \\ \dots & \dots & \dots \\ \sigma_{r_n, s_1}(u), & \dots\dots, & \sigma_{r_n, s_n}(u) \end{vmatrix} \frac{1}{D_n E_n}.$$

and
$$\sigma^n(u)\,\sigma_{12\dots\dots(2n+1)}(u) = \begin{vmatrix} \sigma_{r_1, s_1}(u), & \dots\dots, & \sigma_{r_1, s_{n+1}}(u) \\ \dots & \dots & \dots \\ \sigma_{r_n, s_1}(u), & \dots\dots, & \sigma_{r_n, s_{n+1}}(u) \\ \sigma_{s_1}(u) & , \dots\dots, & \sigma_{s_{n+1}}(u) \end{vmatrix} \frac{1}{D_n E_{n+1}}$$

These expressions bring into evidence the fact that $\sigma_{12\dots\dots k}(u)$ vanishes to order $\frac{1}{2}k$, or $\frac{1}{2}(k+1)$, when the arguments $u_1, \dots\dots, u_p$ are zero, according as k is even or odd; and they show that the first terms in the expansion of $\sigma_{12 \ldots\ldots 2n}(u)$ in powers of $u_1, \ldots\ldots, u_p$, are those given symbolically by

$$\frac{1}{n!}\,\phi(\xi_1)\ldots\ldots\phi(\xi_n)\,\Delta(\xi_1, \ldots\ldots, \xi_n),$$

where $\phi(\xi) = P(\xi)/(\xi - a_1)\ldots\ldots(\xi - a_{2n})$, $\Delta(\xi_1, \ldots\ldots, \xi_n)$ denotes the product of the squares of the differences of $\xi_1, \ldots\ldots, \xi_n$, and, after the divisions have been carried out, we are to replace the powers of the equivalent symbols $\xi_1, \ldots\ldots, \xi_n$ according to the law expressed by

$$\xi_r^{i-1} = u_i, \quad \xi_r^{\circ} = u_1. \qquad \begin{pmatrix} r = 1, 2, \ldots\ldots, n \\ i = 1, 2, \ldots\ldots, p \end{pmatrix}$$

Similarly, with $\phi(\xi) = P(\xi)/(\xi - a_1)\ldots\ldots(\xi - a_{2n+1})$, the first term in the expansion of $\sigma_{12 \ldots\ldots 2n+1}(u)$ is given by

$$\frac{1}{(n+1)!}\,\phi(\xi_1)\ldots\ldots\phi(\xi_{n+1})\,\Delta(\xi_1, \ldots\ldots, \xi_{n+1}).$$

Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442. The function $\sigma_{12 \ldots\ldots 2n}(u)$ has the 'algebraical characteristic' given by

$$\phi_x^{p+1-2\mu} = (x - a)\,\frac{P(x)}{(x - a_1)\ldots\ldots(x - a_{2n})},$$

$$\psi_x^{p+1+2\mu} = Q(x)\,.\,(x - a_1)\ldots\ldots(x - a_{2n}),$$

and the function $\sigma_{12 \ldots\ldots (2n+1)}(u)$ the algebraical characteristic given by

$$\phi_x^{p+1-2\mu} = \frac{P(x)}{(x - a_1)\ldots\ldots(x - a_{2n+1})},$$

$$\psi_x^{p+1+2\mu} = (x - a)\,Q(x)\,(x - a_1)\ldots\ldots(x - a_{2n+1}).$$

VIII.—*New expression for the square of a theta function of three or more suffixes in terms of the squares of functions of one and two suffixes.*

It is known that the skew symmetrical determinant of $2n$ rows and columns, of which the $(i, j)^{\text{th}}$ element $a_{i,j}$ is such that $a_{i,j} = -a_{j,i}$, $a_{i,i} = 0$, is the square of a rational polynomial of degree n in the elements of the determinant; this polynomial—called by Cayley a Pfaffian—we denote by $(123 \ldots . 2n)$; it may be defined by the equation

$$(12 \ldots . 2n) = a_{12}(34 \ldots . 2n) - a_{13}(245 \ldots . 2n) \\ + a_{14}(235 \ldots . 2n) - \ldots . + a_{1,\,2n}(23 \ldots . \overline{2n-1}),$$

where $n = 2, 3, 4, \ldots .$

With this notation we have the theorem expressed by the two following equations

$$\vartheta^{2(n-1)}(u) \,.\, \vartheta^2_{12 \ldots . 2n}(u) = (12 \ldots . 2n),$$

$$\vartheta^{2n}(u) \,.\, \vartheta^2_{12 \ldots . (2n+1)}(u) = (012 \ldots . 2n+1),$$

where, if $b_1, \ldots . , b_{2n+1}$ be any finite branch places, and $B_1 B_2 \ldots . B_k$ denote the sum (without reduction) of the characteristics associated with the half-periods $u^{a,\,b_1}, \ldots . , u^{a,\,b_k}$,

$$\vartheta_{12 \ldots . k}(u) = \vartheta(u \,|\, B_1 B_2 \ldots . B_k),$$

as before, and

$$a_{ij} = \vartheta^2_{ij}(u) \text{ when } i < j, \quad a_{ij} = -\vartheta^2_{ij}(u) \text{ when } i > j,$$

$$a_{0,\,i} = \vartheta^2_i(u), \quad a_{i,\,0} = -\vartheta^2_i(u).$$

These equations enable us to express the quotient $\vartheta^2_{12 \ldots . k}(u)/\vartheta^2(u)$, in which $k > 2$, as a rational integral polynomial in the $\frac{1}{2}p(p+1)$ quotients $\vartheta^2_i(u)/\vartheta^2(u)$, $\vartheta^2_{ij}(u)/\vartheta^2(u)$.

In the applications which are here to be made of these expressions the branch places $b_1, \ldots . , b_{2n+1}$ will be chosen from among $a_1, \ldots . , a_p$.

IX.—*On an addition equation for the hyperelliptic theta functions and the resulting proof of the expressions of the quotients $\vartheta^2(u \,|\, A_i)/\vartheta^2(u)$, $\vartheta^2(u \,|\, A_i A_j)/\vartheta^2(u)$ by means of functions $\wp_{r,\,s}(u)$.*

Let A_i denote the half-integer characteristic associated with the half-period $u^{a,\,a_i}$, and let Q denote any one of the group of 2^p characteristics formed by the addition of $0, 1, 2, \ldots . , p$ of the characteristics $A_1, \ldots . , A_p$. Further, let

$$\wp_{i,\,j}(u) = -\frac{\partial^2}{\partial u_i \, \partial u_j} \log \vartheta(u).$$

Then we have the addition equation (given by Königsberger, Crelle, LXIV)

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)=\sum_{Q}\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q).$$

Now suppose that in this equation, for small values of the arguments v, both sides are expanded in powers of these arguments, and the coefficients of the quadratic powers of these arguments on the two sides of the equation are equated to one another. As follows from No. VII, the only terms on the right side wherein the lowest powers of the arguments v are not of higher degree than the second are those involving functions of one and two suffixes. The left side is equal to

$$\vartheta^2(0)\,\vartheta^2(u)\left[1-\sum_{i=1}^{p}\sum_{j=1}^{p}v_i v_j\,\wp_{ij}(u)+\dots\right].$$

Hence we obtain $\frac{1}{2}p(p+1)$ equations whereby the $\frac{1}{2}p(p+1)$ quotients $\vartheta^2(u\,|\,A_i)/\vartheta^2(u)$, $\vartheta^2(u\,|\,A_iA_j)/\vartheta^2(u)$ are expressible linearly by the functions $\wp_{ij}(u)$.

Utilizing the results of No. VI to solve these equations, we find, if

$$\frac{\vartheta(u)}{\vartheta(0)}=1+\tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}c_{ij}\,u_i\,u_j+\dots,$$

that

$$\frac{\vartheta^2(u\,|\,A_r)}{\vartheta^2(u)}=-\frac{1}{M_r}\sum_{i=1}^{p}\left[c_{p,\,i}+\wp_{p,\,i}(u)\right]a_r^{i-1},$$

$$\frac{\vartheta^2(u\,|\,A_rA_s)}{\vartheta^2(u)}=\frac{a_r\sim a_s}{M_rM_s}\sum_{i=1}^{p}\sum_{j=1}^{p}\left[c_{i,\,j}+\wp_{i,\,j}(u)\right]a_r^{i-1}a_s^{j-1},$$

where, with the meaning explained in No. VI,

$$M_r=i\sqrt{f'(a_r)}/4,\qquad (r=1,\,2,\,\dots,\,p)$$

and $a_r\sim a_s$ is written instead of a_r-a_s to indicate that in the ascending order of the branch places $a_1,\dots,a_p$, a_r has a higher place than a_s, or in other words, that $r>s$.

Results of this form have been given before. See, for instance, Wiltheiss, Math. Annal., XXXI (1888), p. 417, and Bolza, American Journal, XVII (1895), and the references to Brioschi and others there given.

In these equations, and in the previous Nos. of this Analysis, the function $\vartheta(u)$ is that given by

$$\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_p=-\infty}^{\infty} e^{au^2+2\pi i vn+i\pi\tau n^2},$$

where $v_1, \ldots, v_p$ are Riemann's normal integrals of the first kind, the periods of v_r being

$$0, \ldots, 0, 1, 0, \ldots, 0, \quad \tau_{r,1}, \tau_{r,2}, \ldots, \tau_{r,p},$$

vn denotes $v_1n_1 + \ldots + v_pn_p$, τn^2 denotes $\Sigma\Sigma\tau_{ij}n_in_j$, $(i, j = 1, \ldots, p)$, au^2 denotes $\Sigma\Sigma a_{ij}u_iu_j$, $(i, j = 1, 2, \ldots, p)$, wherein $a_{ij} = a_{ji}$, and the $\frac{1}{2}p(p+1)$ quantities $a_{i,j}$ are those occurring in the equation

$$2\sum_{i=1}^{p}\sum_{j=1}^{p} a_{i,j}\, u_i^{x,a} u_j^{z,c} = \Pi_{z,c}^{x,a} - \int_a^x\int_c^z \frac{dx\,dz}{ys}\cdot\frac{2ys+F(x,z)}{4(x-z)^2},$$

where $\Pi_{z,c}^{x,a}$ is Riemann's normal elementary integral of the third kind, and $F(x, z)$ is any symmetrical integral polynomial in x, z, of degree $p+1$ in each, which satisfies the equations

$$F(z, z) = 2f(z), \quad \left[\frac{\partial}{\partial x}F(x, z)\right]_{x=z} = \frac{d}{dz}f(z).$$

The coefficients $c_{i,j}$ are determinable by the fact that

$$\sum_{i=1}^{p}{}'\sum_{j=1}^{p} c_{ij}\,u_iu_j = \frac{4\left[P(\xi_1)\,Q(\xi_2) + P(\xi_2)\,Q(\xi_1)\right] - F(\xi_1, \xi_2)}{4(\xi_1-\xi_2)^2},$$

where the meaning is that after the division on the right-hand has been carried out (as is always possible), we are to put

$$\xi_1^{i-1} = u_i = \xi_2^{i-1}, \quad \xi_1^0 = u_1 = \xi_2^0. \qquad (i = 1, 2, \ldots, p)$$

In particular we may take each of c_{ij} to be zero.

X.—*A fundamental theorem.*

If Q be used, precisely as in No. IX, to denote one of a certain group of 2^p characteristics, then we have the theorem

$$\frac{\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)}$$

$$= \begin{array}{l}\text{rational integral polynomial}\\ \text{in the } p(p+1) \text{ functions}\end{array}\ \frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u\,|\,Q),\ \frac{\partial^2}{\partial v_i\,\partial v_j}\log\vartheta(v\,|\,Q).$$

This follows immediately by combining the results of Nos. VIII and IX.

Two examples may be given; let $\sigma_{12\,\ldots\ldots\,k}(u)$ denote the function defined in No. VII; then

(α) for $p = 2$, if

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}\log \sigma_{12}(u),$$

we have

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + (a_1 + a_2 + c_{22})(\wp_{12}(u) - \wp_{12}(v)) \\ + (a_1a_2 - c_{12})(\wp_{22}(u) - \wp_{22}(v)) + \wp_{21}(u)\,\wp_{22}(v) - \wp_{21}(v)\,\wp_{22}(u).$$

In particular, when, with

$$f(x) = a_x^{2p+2} = 4\,(x^5 + 5A_1x^4 + 10A_2x^3 + 10A_3x^2 + 5A_4x + A_5),$$

we take, as we may,

$$F(x,\,z) = 2a_x^{p+1}a_z^{p+1},$$

we find $\qquad a_1 + a_2 + c_{22} = -2A_1, \quad a_1a_2 - c_{12} = A_2;$

and when, with

$$f(x) = 4x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda,$$

we take, as we may, with $\lambda_5 = 4$, $\lambda_6 = 0$, $\lambda_7 = 0$,

$$F(x,\,z) = \sum_{i=0}^{p+1} x^iz^i\,[2\lambda_{2i} + \lambda_{2i+1}(x+z)],$$

we find $\qquad a_1 + a_2 + c_{22} = 0, \quad a_1a_2 - c_{12} = 0.$

(β). For $p = 3$, if

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i\,\partial u_j}\log \sigma_{123}(u),$$

we have

$$-\frac{\sigma_{123}(u+v)\,\sigma_{123}(u-v)}{\sigma_{123}^2(u)\,\sigma_{123}^2(v)} \\ = [\wp_{31}(u) - \wp_{31}(v)]^2 - [\wp_{33}(u) - \wp_{33}(v)]\,[\wp_{11}(u) - \wp_{11}(v)] \\ + [\wp_{21}(u) - \wp_{21}(v)]\,[\wp_{23}(u) - \wp_{23}(v)] - [\wp_{22}(u) - \wp_{22}(v)]\,[\wp_{31}(u) - \wp_{31}(v)].$$

Equations of this form, in each of which there occurs *only one function,* and its second logarithmic derivatives, appear to be of importance, not only for the theory of theta functions, but also for the general theory of periodic functions of several independent variables.

Of the functions of u occurring on the right side of these equations, every $p+1$ are connected by an algebraical relation; and they can all be expressed rationally by means of $p+1$ functions having the same periods, and so connected. The discussion of these properties is reserved*—here the expressions are given in the form in which they naturally present themselves in the first instance.

ON THE HYPERELLIPTIC SIGMA FUNCTIONS. DEVELOPMENT.

SECTION I.—*Preliminary lemma.*

A certain factorial function in general.

I. On any Riemann surface, dissected to a p-ply connected surface in the ordinary way, by (a)-cuts or period loops, and (b)-cuts or period loops, there exists one and only one single valued function satisfying the conditions, (i) of being infinite to the first order, with a residue equal to $+1$, at an arbitrary place c, (ii) of vanishing to the first order at an arbitrary place z, (iii) that its value shall be the same at any two opposite points of an (a)-cut, while its value at a point on the left or positive side of a (b)-cut, say (b_r), shall be $e^{2\pi i v_r^{z,c}}$ times its value at the corresponding point of the right side, $v_1^{x,a}, \ldots\ldots, v_p^{x,a}$ being Riemann's normal integral of the first kind.

That there cannot be two such functions is clear, since their quotient would be a single-valued function without factors at the (a) and (b) cuts, and without infinities, and having the value unity at the place c.

While, on the other hand, if K be a properly chosen quantity, independent of x, and $\Pi_{z,c}^{x,k}$ be Riemann's normal elementary integral of the third kind, with z and c as infinities, chosen so as to vanish at the place k, the function is clearly given by $Ke^{\Pi_{z,c}^{x,k}}$.

There are reasons however why this form is not convenient; we therefore introduce another. Let

$$\Theta(u) = \Sigma e^{2\pi i n u + i\pi\tau n^2},$$

where nu denotes $n_1u_1 + \ldots\ldots + n_pu_p$, τn^2 denotes $\Sigma\Sigma\tau_{ij}n_in_j$, and the summation is in regard to each of $n_1, \ldots\ldots n_p$ over all integers from $-\infty$ to $+\infty$; further, let $\frac{1}{2}\Omega$ be any *odd* half-period, that is, a set of p simultaneous half-periods, of which one element is given by

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* [*Added, September*, 1898. The reader may refer to a note in the Proceedings of the Cambridge Philosophical Society, May Term, 1898 (vol. IX, Part IX)].

$$\tfrac{1}{2}(\Omega)_i = s_i + \tau_{i,1} s'_1 + \dots\dots + \tau_{i,p} s'_p, \qquad (i = 1, 2, \dots\dots, p)$$

then, assuming as known the fact that the function of x expressed by $\Theta(v^{x,z} + \frac{1}{2}\Omega)$ has p zeros of the first order, one of them at the place z and the others independent of z, and has no infinities, the factorial function in question is clearly given by

$$f(x;\ z,\ c) = \frac{\Theta(v^{x,z} + \frac{1}{2}\Omega)}{\Theta(v^{x,c} + \frac{1}{2}\Omega)} \frac{\sum\limits_{i=1}^{p} (Dv_i)_c \Theta'_i(\frac{1}{2}\Omega)}{\Theta(v^{c,z} + \frac{1}{2}\Omega)},$$

wherein $\sum\limits_{i=1}^{p} (Dv_i)_c \Theta'_i(\frac{1}{2}\Omega)$ represents the value, at the place c, of the differential coefficient of the function $\Theta(v^{x,c} + \frac{1}{2}\Omega)$.

II. For the function $f(x;\ z,\ c)$ we have the property

$$f(z';\ z,\ c) = -f(z;\ z',\ c),$$

namely, the value at any place z', of the function defined in (I), with c as pole and z as zero, is the negative of the value, at the place z, of the corresponding function with the same pole c but a different zero z'.

This is immediately obvious from the representation given, bearing in mind the equation

$$\Theta(v^{x,z} + \tfrac{1}{2}\Omega) = -e^{-2\pi i s' v^{x,z}} \Theta(v^{z,x} + \tfrac{1}{2}\Omega),$$

where $s'v^{x,z}$ is, for abbreviation, put for $s'_1 v_1^{x,z} + \dots\dots + s'_p v_p^{x,z}$. This equation expresses a fundamental property of the function Θ.

III. The proof of the property in (II) can be given in a more elementary way in terms of integrals of the third kind only. For that purpose we require a result contained in the following proposition:

If $P_{z,c}^{x,k}$ represent any elementary integral of the third kind, with infinities at z and c, and chosen so as to vanish at k, and if x, x_1, x_2, x_3 be any places whatever, we have

$$P_{x_2,x_3}^{x,x_1} + P_{x_3,x_1}^{x,x_2} + P_{x_1,x_2}^{x,x_3} = \text{odd integral multiple of } \pi i.$$

To prove this we notice first that it is sufficient to prove it for the case where the integral of the third kind is taken to be Riemann's normal elementary

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integral of the third kind; the more general statement immediately follows from this. Now if (x, z) be used for shortness to denote the function $\Theta(v^{x, z} + \frac{1}{2}\Omega)$, it can be easily proved, by comparing the zeros and poles of the two sides of the equation and the factors at the period-loops, that the following equation holds:

$$\Pi_{z, c}^{x, k} = \frac{[x, z]\,[k, c]}{[x, c]\,[k, z]};$$

hence we have

$$e^{\Pi_{x_2, x_3}^{x, x_1} + \Pi_{x_3, x_1}^{x, x_2} + \Pi_{x_1, x_2}^{x, x_3}} = \frac{[x, x_2][x_1, x_3]}{[x, x_3][x_1, x_2]} \cdot \frac{[x, x_3][x_2, x_1]}{[x, x_1][x_2, x_3]} \; \frac{[x, x_1][x_3, x_2]}{[x, x_2][x_3, x_1]},$$

and, in virtue of the equation $[x, z] = -e^{2\pi i s' v^{z, x}}[z, x]$, already quoted, the right side is equal to -1.

Cor. 1. Putting $x = x_1$ and, for greater clearness, replacing x_1, x_2, x_3 respectively by z, a, b, we have the result

$$limit\left(e^{\Pi_{z, a}^{x, b} - \Pi_{z, b}^{x, a}}\right)_{x=z} = -1.$$

This includes the general form of the proposition.

Cor. 2. In order to use the result of (III) to prove the result of (II), notice that, ξ being an arbitrary place,

$$\frac{f(x;\ z, c)}{f(\xi;\ z, c)} = e^{\Pi_{z, c}^{x, \xi}}, \quad \frac{f(x;\ z', c)}{f(\xi;\ z', c)} = e^{\Pi_{z', c}^{x, \xi}},$$

and therefore

$$\frac{f(z';\ z, c)}{f(z;\ z', c)} = \frac{f(\xi;\ z, c)}{f(\xi;\ z', c)}\, e^{\Pi_{z, c}^{z', \xi} - \Pi_{z', c}^{z, \xi}};$$

while also

$$\frac{f(x;\ z, c)}{f(x;\ z', c)} = e^{\Pi_{z, z'}^{x, c}};$$

thus

$$\frac{f(z';\ z, c)}{f(z;\ z', c)} = e^{\Pi_{z, z'}^{\xi, c} + \Pi_{z', c}^{\xi, z} + \Pi_{c, z}^{\xi, z'}} = -1,$$

which is the result we desired.

This proof of the result in (II) is to be preferred to the former one, because, although the theorem for integrals of the third kind upon which it depends has here been deduced from the properties of theta functions, that theorem is, I think, clearly capable of a more elementary proof by contour integration.

Cor. 3. It is easy to see that

$$f(x;\ z,\ c)\,e^{-\Pi_{z,\ c}^{x,\ z'}}+f(x';\ z',\ c)\,e^{-\Pi_{z',\ c}^{x',\ z}}=0.$$

In case x, x' coincide respectively with z', z, this becomes the proposition in (II).

The results in this Section I are of a preliminary character, and in what follows will only be used for the hyperelliptic case.

SECTION II.

Of methods of dissecting the hyperelliptic Riemann Surface.

The fundamental algebraic equation is taken to be

$$y^2=4\,(x-c_1)(x-a_1)\,\ldots\ldots\,(x-c_p)(x-a_p)(x-c),\ =f(x),$$

wherein $c_1,\ a_1,\ \ldots\ldots,\ c_p,\ a_p,\ c$ are given (complex or real) quantities. Over the x plane we suppose a two-sheeted Riemann surface to be constructed, with $p+1$ cross lines, between the places $(c_1,\ a_1)$, $(c_2,\ a_2)$, $\ldots\ldots$, $(c_p,\ a_p)$, $(c,\ \infty)$. We suppose the branch values and the branch places $c_1,\ a_1,\ \ldots\ldots$ always to be taken in the order

$$c_1,\ a_1,\ c_2,\ a_2,\ \ldots\ldots,\ c_r,\ a_r,\ \ldots\ldots,\ c_p,\ a_p,\ c,\ \infty,$$

and shall often denote them respectively by

$$b_1,\ b_2,\ b_3,\ b_4,\ \ldots\ldots,\ b_{2r-1},\ b_{2r},\ \ldots\ldots,\ b_{2p-1},\ b_{2p},\ b_{2p+1},\ a,$$

using throughout the symbol a for the branch place at infinity. By the *standard case* we mean that in which $c_1,\ a_1,\ \ldots\ldots,\ a_p,\ c$ are real and in ascending order of magnitude; by the difference $b_i \sim b_j$ we mean that in which the b of less suffix is subtracted from that of greater suffix, namely, $i>j$; thus in the standard case $b_i \sim b_j$ will be positive as well as real. We suppose the Riemann surface to be changed into a p-ply connected surface in the ordinary way, by means of p period-loop-pairs, the sides of each of these pairs constituting a closed curve, and speak of the resulting surface as the dissected Riemann surface, using sometimes by analogy the word cuts in place of period loops.

Let $u_1^{x,\ a},\ \ldots\ldots,\ u_p^{x,\ a}$ be any p linearly independent integrals of the first kind, single-valued upon the dissected surface; denote the periods of $u_r^{x,\ a}$ at the first and second of the i^{th} pair of period-loops, namely, at the loops (a_i), (b_i), respectively by $2\omega_{r,i}$, $2\omega'_{r,i}$. Then if b denote any one of the $(2p+1)$ finite branch places, we have

$$u_r^{a,b} = \beta_1\omega_{r,1} + \dots + \beta_p\omega_{r,p} + \beta'_1\omega'_{r,1} + \dots + \beta'_p\omega'_{r,p}, \qquad (r = 1, \dots, p)$$

where $\beta_1, \dots, \beta'_p$ are integers, which are immediately obvious by inspection in case of any specified dissection of the surface; these equations we denote by

$$u^{a,b} = \tfrac{1}{2}\binom{\beta'}{\beta};$$

upon the properties of the $(2p+1)$ half-integer characteristics thus arising depends the definiteness of the determination of the signs of certain square roots arising here.

If Q, K be any two half-integer characteristics given by

$$Q = \tfrac{1}{2}\binom{q'}{q} = \tfrac{1}{2}\binom{q'_1, \dots, q'_p}{q_1, \dots, q_p}, \quad K = \tfrac{1}{2}\binom{k'}{k} = \tfrac{1}{2}\binom{k'_1, \dots, k'_p}{k_1, \dots, k_p},$$

we use the abbreviations

$$\binom{Q}{K} = e^{\pi i q' k} = exp.\ \pi i \sum_{r=1}^{p} q'_r k_r, \quad |Q| = qq' = \sum_{r=1}^{p} q_r q'_r,$$

$$|Q, K| = qk' - q'k = \sum_{r=1}^{p} (q_r k'_r - q'_r k_r),$$

so that

$$e^{\pi i |Q, K|} = \binom{Q}{K}\binom{K}{Q}, \quad e^{\pi i |Q|} = \binom{Q}{Q};$$

further, we denote the characteristic associated with the half-period u^{a, b_i} by B_i; for our purpose the values of the $2p(2p+1)$ quantities

$$\binom{B_i}{B_j}, \quad |B_i, B_j|$$

will be found to be of importance.

It is convenient to choose the dissection of the Riemann surface so that, provided $i > j$, the values of these quantities shall be independent of i and j; we give therefore below examples of methods of dissection whereby

(I) when $i > j$,

$|B_i, B_j| = 1$, $\binom{B_i}{B_j} = 1$, and therefore $|B_j, B_i| = -1$, $\binom{B_j}{B_i} = -1$,

(II) when $i > j$,

$|B_i, B_j| = -1$, $\binom{B_i}{B_j} = 1$, and therefore $|B_j, B_i| = 1$, $\binom{B_j}{B_i} = -1$,

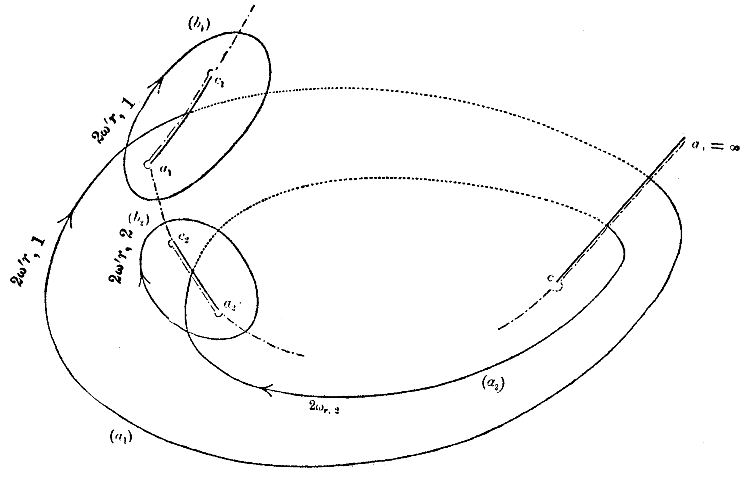
(III) when $i > j$,

$$|B_i, B_j| = \;\; 1, \quad \binom{B_i}{B_j} = -1, \text{ and therefore } |B_j, B_i| = -1, \quad \binom{B_j}{B_i} = \;\; 1,$$

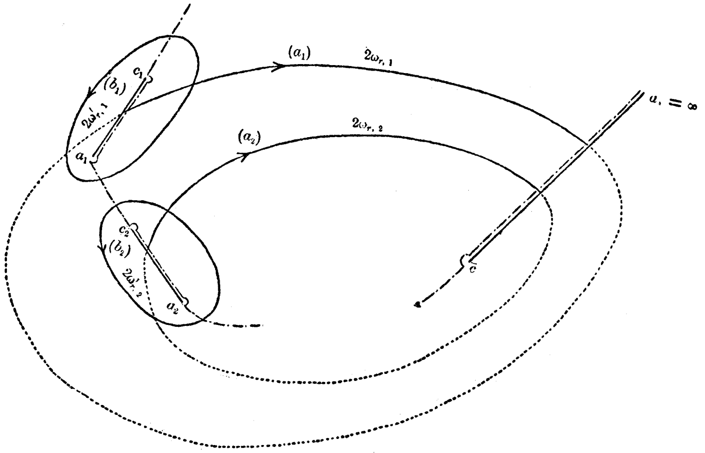
(IV) when $i > j$,

$$|B_i, B_j| = -1, \quad \binom{B_i}{B_j} = -1, \text{ and therefore } |B_j, B_i| = \;\; 1, \quad \binom{B_j}{B_i} = \;\; 1,$$

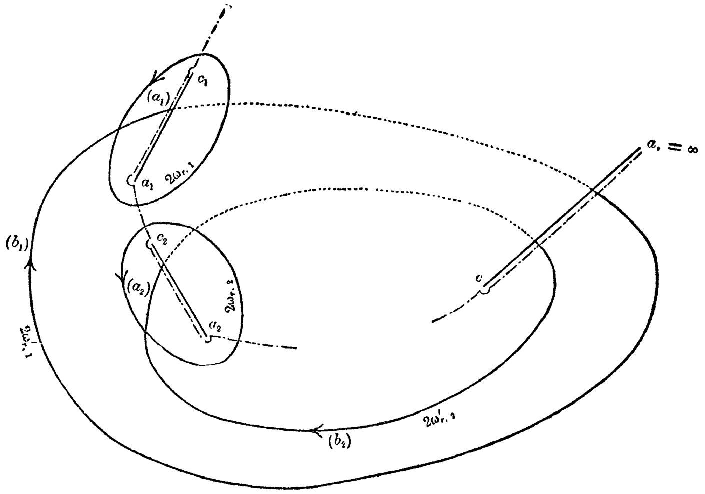
namely, such dissections are represented diagrammatically by the four following figures respectively:



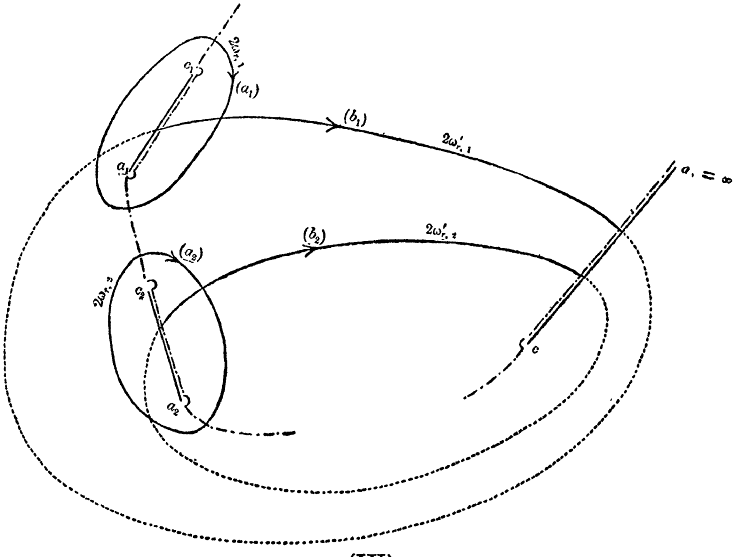
(I)



(II)



(III)



(IV)

We find for these dissections respectively,

$$\text{(I)} \qquad u^{a,\, c_r} = \tfrac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r},$$

$$u^{a,\, a_r} = \tfrac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{r-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{p-r}, \quad u^{a,\, c} = \tfrac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{p},$$

$$\text{(II)}\quad u^{a,\,c_r}=\tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{r-1}\begin{pmatrix}-1\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},$$
$$u^{a,\,a_r}=\tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{r-1}\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},\quad u^{a,\,c}=\tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}^{p},$$
$$\text{(III)}\quad u^{a,\,c_r}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{r-1}\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},$$
$$u^{a,\,a_r}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{r-1}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},\quad u^{a,\,c}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{p},$$
$$\text{(IV)}\quad u^{a,\,c_r}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{r-1}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},$$
$$u^{a,\,a_r}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{r-1}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}^{p-r},\quad u^{a,\,c}=\tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}^{p};$$

from these the facts stated above as to the values of the quantities $|B_i,\, B_j|$, $\begin{pmatrix}B_i\\B_j\end{pmatrix}$ can be immediately verified.

If we denote the periods of an integral $u_r^{x,\,a}$, for the case (I), at the i^{th} period loops of the first and second kind respectively, by $2\omega_{r,\,i}$, $2\omega'_{r,\,i}$, and denote the corresponding periods for the dissections (II), (III), (IV) respectively by

$$[2\omega_{r,\,i}]_2,\ [2\omega'_{r,\,i}]_2\,;\ [2\omega_{r,\,i}]_3,\ [2\omega'_{r,\,i}]_3\,;\ [2\omega_{r,\,i}]_4,\ [2\omega'_{r,\,i}]_4,$$

we immediately find, supposing of course that the values of y are the same, at any the same place of the surface, in all the four cases, that

$$\begin{matrix}[\omega_{r,\,i}]_2=-\omega_{r,\,i}, & [\omega_{r,\,i}]_3=-\omega'_{r,\,i}, & [\omega_{r,\,i}]_4=\omega'_{r,\,i}, \\ [\omega'_{r,\,i}]_2=-\omega'_{r,\,i}, & [\omega'_{r,\,i}]_3=\omega_{r,\,i}, & [\omega'_{r,\,i}]_4=-\omega_{r,\,i},\end{matrix}\qquad\begin{pmatrix}r=1,\,2,\,\ldots\,,p\\ i=1,\,2,\,\ldots\,,p\end{pmatrix}$$

thus the matrices associated with the linear transformation (B. 531) from case (I) to the cases (II), (III), (IV) respectively, are those denoted by

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

It is to be observed that from each of the four dissections (I), (II), (III), (IV) we can immediately derive another by changing the direction of every (a) loop and every (b) loop. Thus, from (1) we can derive a dissection in which the periods are exactly equal to those arising in (II)—in which however the characteristics associated with the half-periods $u^{a,\,c_r}$, $u^{a,\,a_r}$, $u^{a,\,c}$ will have all their elements negative. The quantities $|B_i,\, B_j|$, $\begin{pmatrix}B_i\\B_j\end{pmatrix}$ are, however, unaltered by the alteration of direction of every (a) loop and every (b) loop.

SECTION III.

Of the construction of the functions Θ.

Denoting the characteristic associated with a half-period $u^{a,\, b}$ by $\frac{1}{2}\binom{q'}{q}$ and the matrices of the periods $2\omega_{r,\, i}$, $2\omega'_{r,\, i}$ respectively by 2ω, $2\omega'$, we obtain the normal integrals v and the symmetrical matrix τ by the equations

$$u = 2\omega v, \quad \omega' = \omega\tau;$$

then the function $\Theta(u\,|\,u^{a,\, b})$ or $\Theta(u\,|\,B)$ is given by

$$\sum_n e^{2\pi i v\,(n+\frac{1}{2}q') + i\pi\tau\,(n+\frac{1}{2}q')^2 + i\pi q\,(n+\frac{1}{2}q')},$$

where $v(n+\frac{1}{2}q')$ denotes $\sum_{r=1}^{p} v_r(n_r+\frac{1}{2}q'_r)$, $\tau(n+\frac{1}{2}q')^2$ denotes $\sum_{r=1}^{p}\sum_{s=1}^{p}\tau_{r,\, s}(n_r+\frac{1}{2}q'_r)(n_s+\frac{1}{2}q'_s)$, $q(n+\frac{1}{2}q')$ denotes $\sum_{r=1}^{p} q_r(n_r+\frac{1}{2}q'_r)$, and the summation $\sum_n$ is in regard to each of $n_1, \ldots, n_p$ over all positive and negative integers. It follows that, for the same values of the arguments v, the function is unaffected by the addition of integers to the half-integers $\frac{1}{2}q'_1, \ldots, \frac{1}{2}q'_p$ occurring in the upper line of the characteristic $\frac{1}{2}\binom{q'}{q}$; thus the function $\Theta(u\,|\,u^{a,\, b})$ as defined by the dissection (II) differs from the function $\Theta(u\,|\,u^{a,\, b})$ as defined by the dissection (I) only in the sign of the arguments v; that is, the even functions in (II) are equal to those in (I), but the odd functions have a different sign. Similarly the functions constructed for the dissection (IV) are derived from those for the dissection (I) by a formula given B. 558; the ratio of any function (I) to the corresponding function (IV) consists of a factor independent of the characteristic, multiplied by ε, where ε is 1 in the case of the even functions, and i in the case of the odd functions.

It is clear that the theta functions arising for any one of the four dissections will be unaltered by changing the direction of every (a) loop and every (b) loop; for this change alters the sign of the characteristic and also the sign of the normal integrals.

Further, if $B_r = \frac{1}{2}\binom{q'}{q}$, $B_s = \frac{1}{2}\binom{k'}{k}$ be the two characteristics associated respectively with the half-periods $u^{a,\, b_r}$, $u^{a,\, b_s}$, and

$$q'_r + k'_r = 2M'_r + \lambda'_r, \quad q_r + k_r = 2M_r + \lambda_r, \quad (r = 1, 2, \ldots, p)$$

where λ_r, λ'_r are integers each $= 0$ or 1, and M_r, M'_r are integers, then, by the *reduced* characteristic $B_r B_s$ is meant the characteristic $\frac{1}{2}\binom{\lambda'}{\lambda}$. It is desirable to compare the dissections (I), (II), (III), (IV) in regard to the values of the integers (M, M'); we have the equation

$$\Theta\left[u \,\middle|\, \binom{M'}{M} + \tfrac{1}{2}\binom{\lambda'}{\lambda}\right] = e^{\pi i M\lambda'}\,\Theta\left(u \,\middle|\, \tfrac{1}{2}\binom{\lambda'}{\lambda}\right);$$

hence, denoting the factor $e^{\pi i M\lambda'}$ as *the reduction factor for the places* b_r, b_s, we find that for the dissections

(I) and (II), the reduction factor is -1 for all cases in which the less of the two values b_r, b_s—that is, the one of these occurring first in the ascending order of the $2p+1$ finite branch places—is one of $a_1, \ldots, a_p$, and is otherwise $+1$.

(III) and (IV), the reduction factor is -1 when the places b_r, b_s are such a pair as c_r and a_r, and is otherwise $+1$.

In order to avoid the consideration of the reduction factor we shall frequently in what follows use $\Theta(u \,|\, B_r B_s)$ to denote the function in which the characteristic is *not* reduced, namely the function $\Theta\left[u \,\middle|\, \frac{1}{2}\binom{q'+k'}{q+k}\right]$, and for this purpose we shall reverse the ordinary rule: *Unless the contrary is stated it will be intended in the function* $\Theta(u \,|\, B_r B_s)$ *that the characteristic is unreduced.*

SECTION IV.

Of the fundamental radical functions on the Riemann surface.

On the dissected surface, if b denote any one of the finite branch places, either one of the square roots of $x - b$ is single-valued; this follows from the fact that $x - b$ vanishes to the second order at b and is infinite to the second order at infinity; to fix the value of this square root it is only necessary to specify the sign for some one value of x; we suppose the sign chosen for $x = \infty$, and that this sign is the same for each of the $2p+1$ functions in which b is in turn every one of the finite branch places; supposing that the values of y have been allocated to the places of the surface before dissection, and noticing that the sign of a product of $2p+1$ factors is altered if the sign of every factor is altered, we agree that the signs of the functions considered, which functions are henceforth to be denoted simply by symbols $\sqrt{x-b}$, shall be such that for a point in the lower sheet at infinity, the ratio

$$\tfrac{1}{2}y \Big/ \prod_{r=1}^{2p+1}\sqrt{x-b_r}$$

shall be equal to $+1$; since this ratio is a single-valued continuous function whose square is $+1$, it follows that, for every place of the surface,

$$y = 2\prod_{r=1}^{2p+1}\sqrt{x-b_r}\,;$$

that is, $$y = 2\sqrt{x-c_1}\,\sqrt{x-a_1}\,\ldots\ldots\sqrt{x-a_p}\,\sqrt{x-c}.$$

Further, for the sake of definiteness, we suppose the sign of the infinitesimal at infinity, which is such that $x = t^{-2}$, so chosen that at infinity, for each of the functions $\sqrt{x-b}$,

$$\lim_{t=0}\,(t\sqrt{x-b}) = +1.$$

We may further suppose for real infinite x, and in the lower sheet of the surface, that t is a real *positive* quantity. Thus in the standard case, for all real values of x greater than b, $\sqrt{x-b}$ would be positive as well as real.

Suppose now that, in the abbreviated notation before explained,

$$u^{a,\,b} = \tfrac{1}{2}\begin{pmatrix}\beta'\\ \beta\end{pmatrix},$$

and let $\frac{1}{2}\Omega$ denote an odd half-period—that is, a set of p elements each of the form

$$\tfrac{1}{2}(s_i + \tau_{i,\,1}s'_1 + \ldots\ldots + \tau_{i,\,p}s'_p), \qquad (i = 1,\,2,\,\ldots\ldots,\,p)$$

wherein $s_1, \ldots\ldots, s_p, s'_1, \ldots\ldots, s'_p$ are integers; then, on the dissected surface, the function

$$\sqrt{x-b}\,\frac{\Theta(v^{x,\,a}+\frac{1}{2}\Omega)}{\Theta(v^{x,\,b}+\frac{1}{2}\Omega)}\;\frac{\Theta(v^{a,\,b}+\frac{1}{2}\Omega)}{\sum\limits_{i=1}^{p}(Dv_i)_a\,\Theta'_i(\frac{1}{2}\Omega)}\,e^{-\pi i\beta' v^{x,a}},$$

wherein the second factor of the denominator denotes the value at infinity of the differential coefficient in regard to the infinitesimal of the function $\Theta(v^{x,a}+\frac{1}{2}\Omega)$, is single-valued and continuous, and has no zeros or infinities; at the r^{th} period loops respectively of the first and second kind the square of the function has the factors

$$(e^{-\pi i\beta'_r})^2, \quad (e^{2\pi i v_r^{a,b}-\pi i\tau_r\beta'})^2 = (e^{\pi i\beta_r})^2,$$

each of which is equal to unity; thus the square of the function, being equal to $+1$ at the place infinity of the surface, is everywhere equal to $+1$. Thus the function itself, which is also equal to $+1$ at infinity, is also everywhere equal to $+1$. Hence, in the notation employed in Section I, we have

$$\sqrt{x-b} = e^{\pi i\beta' v^{x,a}} f(x;\, b,\, a),$$

and this result is independent of the method of dissection of the surface.

It follows that the factors of $\sqrt{x-b}$ at the r^{th} period loops respectively of the first and second kind are

$$e^{\pi i\beta'_r}, \quad e^{\pi i\tau_r\beta'+2\pi i v_r^{b,a}} = e^{\pi i\beta_r};$$

this fact we shall denote by

$$\sqrt{x-b} \propto \binom{\beta}{\beta'},$$

and it will be found that this relation may be supposed to express another fact also. The factors of $\sqrt{x-b}$ at the period loops are immediately obvious geometrically by considering the number of revolutions about the branch-place b involved in taking the function round the period loops. The analytical proof has been preferred because it brings into greater prominence the fact that the result obtained is the same for any method of dissection.

Let now d be another finite branch-place such that

$$u^{a,d} = \tfrac{1}{2}\binom{\delta'}{\delta};$$

let the signs $\sqrt{d-b}$, $\sqrt{b-d}$ denote respectively the value of the function $\sqrt{x-b}$ at d, and the value of the function $\sqrt{x-d}$ at b; then it follows from Section I that

$$\sqrt{d-b}\, e^{-\pi i\beta' v^{d,a}} = -\sqrt{b-d}\, e^{-\pi i\delta' v^{b,a}},$$

and therefore

$$\frac{\sqrt{d-b}}{\sqrt{b-d}} = -e^{\frac{1}{2}\pi i[\delta'(\beta+\tau\beta')-\beta'(\delta+\tau\delta')]} = -e^{-\frac{1}{2}\pi i\,|D,\,B|},$$

where $B=\frac{1}{2}\binom{\beta'}{\beta}$, $D=\frac{1}{2}\binom{\delta'}{\delta}$; by squaring it follows that for *every method of dissection*

$$|B,\,D|\equiv 1, \qquad (\text{mod. } 2)$$

and therefore that, also for any method of dissection,

$$\frac{\sqrt{d-b}}{\sqrt{b-d}} = e^{\frac{1}{2}\pi i\,|D,\,B|}.$$

Hence, *in all methods of dissection for which, when* $i>j$, $|B_i,\,B_j|\equiv 1$ (*mod.* 4), *we have* $\sqrt{b_j-b_i}=-i\sqrt{b_i-b_j}$; *in all methods of dissection for which, when* $i>j$, $|B_i,\,B_j|\equiv -1$ (*mod.* 4), *we have* $\sqrt{b_j-b_i}=+i\sqrt{b_i-b_j}$.

With more precise conventions we may use the result somewhat differently, namely, if for two half-integer characteristics $Q=\frac{1}{2}\binom{q'}{q}$, $K=\frac{1}{2}\binom{k'}{k}$, the symbol $\sqrt{\binom{Q}{K}}$ denote $e^{\frac{1}{2}\pi i q'k}$, we have $\sqrt{\binom{D}{B}}\sqrt{d-b}=\sqrt{\binom{B}{D}}\sqrt{b-d}$.

In what follows *any one of the* $p(2p+1)$ *signs* $\sqrt{b_i-b_j}$ *will be used to mean the value of the function* $\sqrt{x-b_j}$ *when* x *is at* b_i; and, in accordance with the established practice in the elliptic case (Schwarz, "Formeln u. Lehrsätze," p. 24; Halphen, "Fonct. Ellipt.," I, p. 192), we shall take for the normal method of dissection one in which, for $i>j$, $|B_i,\,B_j|\equiv 1$ (mod. 4), as in the methods (I), (III) above (pp. 320 and 321); such a method may be described as a negative method, those in which $|B_i,\,B_j|\equiv -1$ (mod. 4) being described as positive methods of dissection.

It is easy to see geometrically, in the figures (I) and (III), in what way the negative sign arises; let a continuous line be drawn in the lower sheet of the surface from $+\infty$ to $-\infty$—indicated by the dotted line in the figures—so as to pass near all the cross lines of the surface; in reaching a branch-place this line may be supposed to describe a semicircle in the *clockwise* direction; thus the description of the semicircle near a branch-place b gives for the function $\sqrt{x-b}$ a factor $-i$. In the dissections (II), (IV) the corresponding semicircles are described in the counterclockwise direction, and give a factor $+i$ for $\sqrt{x-b}$ as the describing point passes round the branch-place b.

The question may arise whether the method of proof just given, depending on the theta functions, may not be vitiated in the hyperelliptic case owing to the occurrence of even functions which vanish for zero values of the arguments; it is for this reason we have, in Section I, given an alternative proof, depending on the fact that if $P_{x_1, x_2}^{x, a}$ denote an elementary integral of the third kind, with its infinities at x_1, x_2, then

$$P_{x_1, x_2}^{x, x_3} + P_{x_2, x_3}^{x, x_1} + P_{x_3, x_1}^{x, x_2} = \text{odd integral multiple of } \pi i;$$

though we have deduced this property by means of the theta functions, it appears clear that it is capable of an elementary proof.

We shall for convenience introduce, beside the function denoted by $\sqrt{x-b}$, a function denoted by $\sqrt{b-x}$, and defined, for every one of the finite branch-places b, by the equation, holding for every value of x,

$$\sqrt{b-x} = +i\sqrt{x-b},$$

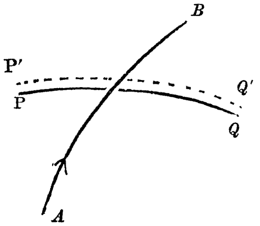
but some care must be exercised in regard to it; the value

$$(\sqrt{b_i-x})_{x=b_j}, \; = (i\sqrt{x-b_i})_{x=b_j} = i\sqrt{b_j-b_i} =, \; ie^{\frac{1}{2}\pi i\,|B_j,\,B_i|}\sqrt{b_i-b_j},$$

is equal to $\sqrt{b_i-b_j}$ only when $|B_i,\ B_j| \equiv 1$ (mod. 4); that is, if for $i>j$, $|B_i,\ B_j| \equiv 1$ (mod. 4), *$\sqrt{b_i-b_j}$ can be interpreted as the value of $\sqrt{b_i-x}$ when x is at b_j only if, in the ascending order of the branch-places, b_i has a higher place than b_j.* It is for this reason we introduce, and shall retain, the convention of using $\sqrt{b_i-b_j}$ only as the value of $\sqrt{x-b_j}$ when x is at b_i.

The function $\sqrt{x-b}$, when x is near to b, has opposite signs in the conjugate places of the Riemann surface—as is obvious by considering that we pass from one of these places to the other by a path going once round the branch-place b. But for values of x which are near to any other finite branch-place, the conjugate values of $\sqrt{x-b}$ are not different, since the function would otherwise vanish at this branch-place. The question then arises of specifying the range of values of x for which the conjugate values of the function $\sqrt{x-b}$ are the same; knowing the factors of $\sqrt{x-b}$ at the period loops, it is easy to do this.

Let AB be a portion of a period loop, P, Q being two places on opposite



sides of this loop, in the same sheet as the loop, P being on the left side of the loop; let P', Q' be their conjugate places; let ε be the factor of $\sqrt{x-b}$ at this period loop, $\zeta_Q\ (=\pm 1)$ be the ratio of the values of $\sqrt{x-b}$ at Q and Q', and ζ_P the ratio of the values at P and P', then we clearly have $\zeta_P=\varepsilon\zeta_Q$. By this remark it is easy to prove the following rule: Let the period loops be projected upon the plane below the Riemann surface; the projections will form a network of closed curves which, for the moment, we may distinguish, from the period loops themselves, as *barriers;* let the conjugate places of the Riemann surface which project, on to this plane, into a point lying within the barrier arising from any period loop, be called the places *within* this period loop; then, *if in the relation*

$$\sqrt{x-b}\propto\binom{\beta}{\beta'},$$

formerly employed, there occurs a unity in the r^{th} *place of the upper line of the symbol on the right, and a zero in the* r^{th} *place of the lower line, it means that the values of* $\sqrt{x-b}$ *are the same for the conjugate places within the loop* (b_r); *if there occurs a unity in the* r^{th} *place of the lower line of the symbol on the right, and a zero in the* r^{th} *place of the upper line, it means that the values of* $\sqrt{x-b}$ *are the same for the conjugate places within the loop* (a_r); *if there occurs a unity in the* r^{th} *place of both the upper and the lower lines of the symbol on the right, it means that the values of* $\sqrt{x-b}$ *are the same within the loop* (a_r), *and also within the loop* (b_r), *except for that region which is within both* (a_r) *and* (b_r), *where the conjugate values of* $\sqrt{x-b}$ *are different. For all regions not specified in this enumeration the conjugate values are different.*

For instance, if we take the dissections (I) or (II) and denote the barriers which are the projections of the period loops (a_r) and (b_r) respectively by (a_r') and (b_r'), the function $\sqrt{x - c_r}$ has its conjugate values equal for all values of x within the non-intersecting closed curves formed by the barriers (a_r'), (b_1'), ..., (b_{r-1}'); for other values of x its conjugate values are opposite;

the function $\sqrt{x - a_r}$ has its conjugate values equal within the barriers (a_r'), (b_1'), ..., (b_r'), save in the region common to the closed curves (a_r') and (b_r'), where they are opposite; for values of x not enclosed by any of these barriers (a_r'), (b_1'), ..., (b_r'), the values of $\sqrt{x - a_r}$ are opposite;

the function $\sqrt{x - c}$ has its conjugate values opposite except within the closed curves formed by the barriers (b_1'), ..., (b_p').

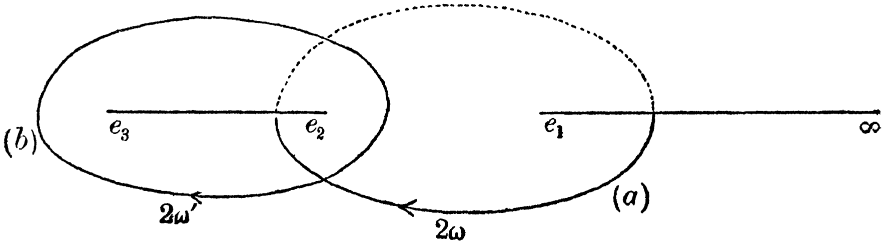
These facts follow from the values of the half-periods $u^{a,\, c_r}$, $u^{a,\, a_r}$, $u^{a,\, c}$ given on pp. 322 and 323. The corresponding results for the dissections (III) and (IV) can be obtained in the same way, or are geometrically obvious from these.

To every function $\sqrt{x - b}$ there are then certain characteristic period loops, namely, those at which its factor is -1. On the Riemann surface the function $\sqrt{x - b}$ does not take every value, for it can take the values $\pm B$ only when $x = b + B^2$, and it may happen that for this value of x the conjugate values of $\sqrt{x - b}$ are the same. If, however, we take two precisely similar and equal exemplars of the Riemann surface and make opposite conventions for the sign of $\sqrt{x - b}$ on these two surfaces, so that at any two corresponding points, one on each surface, the values of $\sqrt{x - b}$ are equal and opposite, and join these two exemplars into one surface by converting the characteristic period loops of the function $\sqrt{x - b}$ into cross lines between the two surfaces, we shall obtain a surface of four sheets whereon $\sqrt{x - b}$ is continuous and single-valued, and takes every value twice. The behavior of $\sqrt{x - b}$ on either of the exemplars is such that if the values of $\sqrt{x - b}$ arising thereon be mapped upon a plane, the plane will be covered once with the exception of k holes, corresponding to the points of which no values of $\sqrt{x - b}$ arise—k being the number of characteristic loops belonging to $\sqrt{x - b}$.

We have proved that the product of the $2p + 1$ functions $\sqrt{x - b}$ is $\frac{1}{2} y$. It follows that for every value of x there is an even number, or none, of these functions, of which the conjugate values are the same.

APPENDIX I TO SECTION IV. *Application to the Elliptic case.*

Replacing the letters c_1, a_1, c which have denoted the branch places, respectively by e_3, e_2, e_1, and taking a dissection of the type (I), we have



$$u^{\infty,\,e_1} = \omega = \tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}, \quad u^{\infty,\,e_2} = \omega + \omega' = \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}, \quad u^{\infty,\,e_3} = \omega' = \tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix};$$

also

$\sqrt{x - e_1}$, $\propto \begin{pmatrix}1\\0\end{pmatrix}$, has factor -1 at loop (b); its conjugate values inside (b) are equal,

$\sqrt{x - e_2}$, $\propto \begin{pmatrix}1\\1\end{pmatrix}$, has factor -1 at loops (a) and (b); its conjugate values are the same inside (a) and inside (b) except for values of x inside both (a) and (b), where they are opposite,

$\sqrt{x - e_3}$, $\propto \begin{pmatrix}0\\1\end{pmatrix}$, has factor -1 at loop (a); its conjugate values inside (a) are the same.

The values of $u^{x,\,\infty}$, $= \int_\infty^x \frac{dx}{y}$, in the two sheets of the surface, (i) outside both the loops (a) and (b) are, for conjugate places, u and $-u$; (ii) inside loop (a), at conjugate places, are u and $-u - 2\omega$; (iii) inside both the loops (a) and (b), at conjugate places of the surface, are u and $-u - 2\omega - 2\omega'$; (iv) inside the loop (b), at conjugate places, are $-u - 2\omega'$; hence putting

$$\chi_1(u) = \frac{\sigma_1(u)}{\sigma(u)} = \sqrt{x - e_1}, \quad \chi_2(u) = \sqrt{x - e_2}, \quad \chi_3(u) = \sqrt{x - e_3},$$

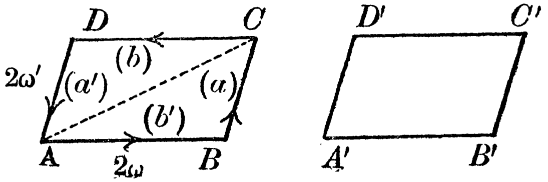
the statements in regard to the three radical functions are easily seen to be summed up in the statements that $\chi_1(u)$, $\chi_2(u)$, $\chi_3(u)$ are odd functions with the respective periods, (i) for $\chi_1(u)$, 2ω, $4\omega'$; (ii) for $\chi_2(u)$, 4ω, $2\omega + 2\omega'$;

(iii) for $\chi_3(u)$, 4ω, $2\omega'$. These facts can be immediately verified by the ordinary formulæ of elliptic functions.

Further, on the surface thus dissected the function

$$\operatorname{sn}(u\sqrt{e_1-e_3}),\ =\sqrt{e_1-e_3}/\sqrt{x-e_3}$$

does not take all values, because the values of the argument u and $-u-2\omega$ give the same values for the function. To get all values we take two exemplars of the Riemann surface, defining the sign of $\sqrt{x-e_3}$ differently in these, and join them into one surface by making the loop (a) a cross line between them, so that the left edge of this loop in one exemplar becomes continuous with the right edge of this loop in the other exemplar, and *vice versa.*



In other words, if we take as the fundamental surface a parallelogram with 2ω, $2\omega'$ as contiguous sides, so that the opposite sides AD, BC represent respectively the right and left edges of the period loop (a), the function $\operatorname{sn}(u\sqrt{e_1-e_3})$ does not take all values within this parallelogram because the values in the triangle ACD are equal to the values in the triangle CAB; to get all values we must take another equal parallelogram wherein the value of $\operatorname{sn}(u\sqrt{e_1-e_3})$ at any place is the negative of the value of $\operatorname{sn}(u\sqrt{e_1-e_3})$ at the corresponding place of the first parallelogram, and then join BC with $A'D'$ and AD with $B'C'$. We thus get the ordinary period parallelogram with $2\omega'$ and 4ω as sides, in which $\operatorname{sn}(u\sqrt{e_1-e_3})$ takes every value twice over. We may then also join $DCD'C'$ with $ABA'B'$, so obtaining an anchor ring upon which $\operatorname{sn}(u\sqrt{e_1-e_3})$ is single-valued, but takes every value twice over.

Another point may be noticed in this elliptic case; similar remarks apply in general; taking the dissection as before, and supposing e_1, e_2, e_3 to be real, and further supposing that for real large positive values of x, in the lower sheet of the surface, the functions $\sqrt{x-e_1}$, $\sqrt{x-e_2}$, $\sqrt{x-e_3}$ are positive, as well as real, then the half-period

$$\omega=\int_{e_1}^{\infty}\frac{dx}{y}=\int_{e_1}^{\infty}\frac{dx}{2\sqrt{x-e_1}\sqrt{x-e_2}\sqrt{x-e_3}}$$

is *real and positive*, while the half-period

$$\omega' = \int_{e_2}^{e_1} \frac{dx}{y}$$

is $+\, i$ *times a real positive* quantity; thus $R\left(\frac{\omega'}{i\omega}\right)$ is positive, while, as we have seen, $\sqrt{e_3 - e_1} = -\, i\sqrt{e_1 - e_3}$, etc. If, however, we had taken a dissection such as (II) we should have obtained $R\left(\frac{\omega'}{i\omega}\right) =$ negative, and, correspondingly, $\sqrt{e_3 - e_1} = +\, i\sqrt{e_1 - e_3}$, etc.

APPENDIX II TO SECTION IV.—*On a certain prime function.*

If $\frac{1}{2}\Omega$ be an odd half-period such that (cf. B. 302)

$$\tfrac{1}{2}\Omega = \tfrac{1}{2}(s + \tau s') = v^{a_p,\, a} - v^{n_1,\, a_1} - \ldots\ldots - v^{n_{p-1},\, a_{p-1}},$$

then (B. 427) we have, in the function,

$$\varpi(x, z) = \frac{\Theta\left(v^{x,\, z} + \frac{1}{2}\Omega\right) e^{\pi i s' v^{x,\, z}}}{\sqrt{(x - n_1)\ldots\ldots(x - n_{p-1})(z - n_1)\ldots\ldots(z - n_{p-1})}} = C\, \frac{(x - z)\, e^{\frac{1}{4}\Pi_{x,\, z}^{\bar{x},\, \bar{z}}}}{\sqrt{ys}}$$

wherein C is a certain constant, and $\bar{x}$, $\bar{z}$ denote the places conjugate to x and z, a function which, as appears from the first form ($n_1, \ldots\ldots, n_{p-1}$ being branch places), is single-valued on the dissected Riemann surface, and has no infinities; the function vanishes to the first order when x is at z, and to order $p - 1$, as a function of x, when x is at infinity. Replacing any factor $x - n_i$ by $x_1 - n_i x_2$, where $x = x_1/x_2$, we have a prime *form*, with no infinities, and only one zero, at $x = z$. We consider, however, only the function. After what has been given, it is easy to determine the factors of this, regarded as a function of x, at the period loops. For if

$$v^{a,\, n_r} = \tfrac{1}{2}(\alpha_r + \tau\alpha_r'),$$

so that

$$\sqrt{x - n_r} \propto \begin{pmatrix} \alpha_r \\ \alpha_r' \end{pmatrix},$$

then

$$\tfrac{1}{2}\Omega = v^{a,\, n_1} + \ldots\ldots + v^{a,\, n_{p-1}} - v^{a,\, a_1} - \ldots\ldots - v^{a,\, a_p}$$
$$= \tfrac{1}{2}\begin{pmatrix} \Sigma\alpha_r' \\ \Sigma\alpha_r \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1, & 1 & , \ldots\ldots, & 1 \\ p, & p - 1 & , \ldots\ldots, & 1 \end{pmatrix},$$

where the dissection is supposed to be of the kind (I); hence

$$s = \Sigma\alpha_r - (p, p-1, \ldots\ldots, 1), \quad s' = \Sigma\alpha_r' - (1, 1, \ldots\ldots, 1).$$

Thus the factor at the k^{th} (a) loop

$$= \frac{e^{\pi i s_k'}}{e^{\pi i \Sigma\alpha_r'}}, = -1;$$

and the factor at the k^{th} period loop of the second kind is

$$\frac{e^{-2\pi i (v_k^{x,z} + \frac{1}{2}\tau_{k,k})} e^{-\pi i s_k}}{e^{\pi i \Sigma\alpha_r}} = (-1)^{p-k+1} e^{-2\pi i (v_k^{x,z} + \frac{1}{2}\tau_{k,k})}.$$

In particular we may take $n_1, n_2, \ldots\ldots, n_{p-1} = a_1, a_2, \ldots\ldots, a_{p-1}$, and $\frac{1}{2}\Omega = u^{a, a_p}$; then, with the dissection (I), *the function of x given by*

$$\varpi(x, z) = \frac{\Theta\left(v^{x,z} \,\middle|\, \frac{1}{2}\begin{pmatrix} 0 & 0 & \ldots\ldots & 0 & 1 \\ 1 & 1 & \ldots\ldots & 1 & 1 \end{pmatrix}\right)}{\sqrt{(x-a_1)\ldots\ldots(x-a_{p-1})}\sqrt{(z-a_1)\ldots\ldots(z-a_{p-1})}}$$

has the following properties: (i) *it vanishes when the place x is at the place z, to the first order, and vanishes $p-1$ times at infinity;* (ii) *it has no infinity;* (iii) $\varpi(x, z) = -\varpi(z, x)$; (iv) *at every loop of the first kind it has the factor* -1; *at the k^{th} loop of the second kind it has the factor*

$$(-1)^{p-k+1} e^{-2\pi i (v_k^{x,z} + \frac{1}{2}\tau_{k,k})}.$$

In the elliptic case the function is the odd theta function; in case $p = 2$ it is

$$\Theta\left(v^{x,z} \,\middle|\, \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right) \Big/ \sqrt{(x-a_1)(z-a_1)}.$$

SECTION V.

On the construction of the function ϑ.

If $R_{z,c}^{x,k}$ be an elementary integral of the third kind, with z, c as infinities, chosen so as to vanish at the place k, and be further such that $R_{z,c}^{x,k} = R_{x,k}^{z,c}$, then there will be an equation of the form

$$R_{z,c}^{x,k} = \prod\nolimits_{z,c}^{x,k} - 2\sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij} u_i^{x,k} u_j^{z,c},$$

wherein $\prod_{z,c}^{x,k}$ is Riemann's normal elementary integral of the third kind,

$u_1^{x,k}, \ldots, u_p^{x,k}$ are any linearly independent integrals of the first kind, and the coefficients $a_{i,j}$ are constants such that $a_{i,j} = a_{j,i}$. Taking then arguments $u_1, \ldots, u_p$, of which the periods of u_r are $2\omega_{r,1}, \ldots, 2\omega_{r,p}, 2\omega'_{r,1}, \ldots, 2\omega'_{r,p}$, these being supposed to satisfy the necessary relations (B. 197, 285), and determining the arguments $v_1, \ldots, v_p$ and the matrix τ by the equations denoted by

$$u = 2\omega v, \quad \omega' = \omega\tau,$$

the function $\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)$ is given by

$$\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right) = \Sigma e^{au^2 + 2\pi i v(n+\frac{1}{2}q') + i\pi\tau(n+\frac{1}{2}q')^2 + i\pi q(n+\frac{1}{2}q')},$$

wherein $au^2 = \Sigma\Sigma a_{ij} u_i u_j$, etc., and the summation is in regard to each of $n_1, \ldots, n_p$ from $-\infty$ to $+\infty$. Thus the function is equal to $e^{au^2}\Theta\left(v \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)$. Associated with the functions $\vartheta(u)$ there are, beside the periods 2ω, $2\omega'$, certain other quantities 2η, $2\eta'$ which are the negatives of the periods of the integrals of the second kind $L_1^{x,a}, \ldots, L_p^{x,a}$ which arise (B. 194) in the equation

$$R_{x,c}^{x,k} = \int_k^x \frac{dx}{2y}\left(\frac{y+s}{x-z} - \frac{y+d}{x-c}\right) + \sum_{i=1}^{p} u_i^{x,k} L_i^{z,c},$$

wherein (z, s), (c, d) are the values of (x, y) when x is at the places z, c respectively.

The general form of $R_{z,c}^{x,k}$ in the hyperelliptic case is

$$R_{z,c}^{x,k} = \int_k^x \int_c^z \frac{dx}{y} \cdot \frac{dz}{s} \cdot \frac{2ys + F(x, z)}{4(x-z)^2},$$

where $F(x, z)$ is any rational integral polynomial, symmetrical in x, z, of order $p+1$ in each, which satisfies the two conditions (B. 315)

$$F(z, z) = 2f(z), \quad \left[\frac{\partial}{\partial x} F(x, z)\right]_{x=z} = \frac{d}{dz} f(z);$$

in case the fundamental algebraic equation be

$$y^2 = \lambda + \lambda_1 x + \ldots + \lambda_{2p+1} x^{2p+1} + \lambda_{2p+2} x^{2p+2} = f(x),$$

it can easily be verified that a possible form for the function $F(x, z)$ is that given by

$$f(x, z) = \sum_{i=0}^{p+1} x^i z^i \left[2\lambda_{2i} + \lambda_{2i+1}(x+z)\right];$$

and any other form for $F(x, z)$ differs from $f(x, z)$ by an expression of the form

$$(x-z)^2(x, z)_{p-1},$$

wherein $(x, z)_{p-1}$ denotes an integral symmetrical polynomial of order $p-1$ in each of x and z; conversely, the $\frac{1}{2}p(p+1)$ coefficients in $(x, z)_{p-1}$ are the arbitrary constants of the integral $R^{x,\,k}_{z,\,c}$; by the suitable choice of these the quantities a_{ij} in the definition of the functions ϑ may be made to take any assigned values.

Another possible form for the function $F(x, z)$, than the one $f(x, z)$ above given, is that given by

$$4\,[P(x)\,Q(z) + P(z)\,Q(x)],$$

where

$$P(x) = (x-a_1)\ldots\ldots(x-a_p), \quad Q(x) = (x-c_1)\ldots\ldots(x-c_p)(x-c).$$

Still another form for $F(x, z)$, which is of great importance, may be most shortly defined thus: in the equation $y^2 = f(x)$ put $x = x_1/x_2$, and write it in the form $y^2x_2^{2p+2} = a_x^{2p+2}$, the notation on the right being the ordinary symbolical notation for binary forms; then a possible form for $F(x, z)$ is that given by $2a_x^{p+1}a_z^{p+1}/x_2^{p+1}z_2^{p+1}$. The advantage of this form is that the resulting integral $R^{x,\,k}_{z,\,c}$ is covariantive under cogredient linear substitutions of the variables x, z (cf. for example Klein, Math. Annal., XXVII (1886), p. 457).

In what follows we shall, unless the contrary is stated, suppose the arbitrary coefficients in the polynomial $F(x, z)$ to be left undetermined. Writing then the expansion

$$\frac{\vartheta(u)}{\vartheta(0)} = 1 + \sum_{i=1}^{p}\sum_{j=1}^{p} c_{i,\,j}\,u_i u_j + \ldots\ldots,$$

we may suppose the coefficients $c_{i,\,j}$ to be at our disposal.

SECTION VI.

The expression of the quotients of the ϑ functions with one suffix by means of algebraic functions.

In this section we suppose the Riemann surface dissected in one of the ways (I), (II), (III), (IV). We shall however ultimately adopt (I) as the normal method of dissection. We use the following notations, other than those already introduced:

$$P(x) = (x - a_1) \dots\dots (x - a_p), \quad Q(x) = (x - c_1) \dots\dots (x - c_p)(x - c);$$

$$y^2 = 4P(x)\,Q(x), \ = f(x);$$

$$F(x) = (x - x_1)(x - x_2) \dots\dots (x - x_p);$$

$$u_i = u_i^{x_1, a_1} + u_i^{x_2, a_2} + \dots\dots + u_i^{x_p, a_p}, \qquad (i = 1, 2, \dots\dots, p)$$

wherein

$$u_i^{x, k} = \int_k^x \frac{x^{i-1}dx}{y};$$

$$V_i = V_i^{x_1, a_1} + V_i^{x_2, a_2} + \dots\dots + V_i^{x_p, a_p}, \qquad (i = 1, 2, \dots\dots, p)$$

where

$$V_i^{x, k} = \frac{\sqrt{f'(a_i)}}{P'(a_i)} \int_k^x \frac{P(x)}{x - a_i} \frac{dx}{2y}$$

and

$$\sqrt{f'(a_i)} = 2\sqrt{a_i - c_1}\sqrt{a_i - a_1}\sqrt{a_i - c_2} \dots\dots \sqrt{a_i - c},$$

$$P'(a_i) = (a_i - a_1)(a_i - a_2) \dots\dots (a_i - a_p);$$

then

$$u_i^{x, k} = \frac{2a_1^{i-1}}{\sqrt{f'(a_1)}} V_1^{x, k} + \dots\dots + \frac{2a_p^{i-1}}{\sqrt{f'(a_p)}} V_p^{x, k},$$

and

$$\frac{\partial}{\partial V_i} = \frac{2}{\sqrt{f'(a_i)}} \left(\frac{\partial}{\partial u_1} + a_i \frac{\partial}{\partial u_2} + \dots\dots + a_i^{p-1} \frac{\partial}{\partial u_p} \right),$$

$$(i = 1, 2, \dots\dots, p).$$

Further, putting

$$P(x) = x^p + h_1 x^{p-1} + \dots\dots + h_p,$$

we have

$$\frac{P(x)}{x - a_r} = x^{p-1} + (a_r + h_1)\, x^{p-2} + \dots\dots + (a_r^{p-1} + h_1 a_r^{p-2} + \dots\dots + h_{p-1})$$

$$= x^{p-1} + \chi_1(a_r)\, x^{p-2} + \dots\dots + \chi_{p-1}(a_r), \text{ say,}$$

so that

$$a_i^{p-1} + \chi_1(a_r)\, a_i^{p-2} + \dots\dots + \chi_{p-1}(a_r) = 0 \text{ or } P'(a_r),$$

according as $i \neq r$, or $i = r$;

thus

$$u_p + \chi_1(a_r)\, u_{p-1} + \dots\dots + \chi_{p-1}(a_r)\, u_1 = 2V_r \frac{P'(a_r)}{\sqrt{f'(a_r)}};$$

we shall, immediately, introduce a constant λ_r, given, for a proper determination of the sign of the denominator, by

$$\lambda_r = \frac{(-1)^{p-r}\sqrt{P'(a_r)}}{(i\sqrt{f'(a_r)}/4)^{\frac{1}{2}}}\,;$$

with this we have

$$u_p + \chi_1(a_r)\,.\,u_{p-1} + \ldots\ldots + \chi_{p-1}(a_r)\,.\,u_1 = i\lambda_r^2 V_r.$$

If λ, μ, ... denote either linear functions $x - b$, $b - x$, or differences such as $b_i - b_j$, between values of x at the finite branch places, we mean by $\sqrt{\lambda\mu\ldots\ldots}$ the product $\sqrt{\lambda}\sqrt{\mu}\ldots\ldots$, where $\sqrt{\lambda}$, $\sqrt{\mu}$, ... have the values previously assigned to them. We have already introduced $\sqrt{f'(a_r)}$ defined according to this rule.

If, in the variable V_i, we suppose $x_1, \ldots\ldots, x_p$ to be respectively near to the branch places $a_1, \ldots\ldots, a_p$, and put $x_i = a_i + t_i^2$, we easily find that $V_i^{x_r,\,a_r}$ vanishes to the third order when $r \neq i$, while $V_i^{x_i,\,a_i} = t_i$; hence when $x_1, \ldots\ldots, x_p$ come to $a_1, \ldots\ldots, a_p$, the variable $V_i = t_i$.

For the integrals V_i see B. 169; they are more convenient for our purpose than Weierstrass's integrals (B. 325–6).

We find the following equations to hold for the dissection (I):

$$\begin{aligned}
&\sqrt{(-1)^{2p-2r+1}f'(a_r)} = i^{2p-2r+1}\sqrt{f'(a_r)}, &&\sqrt{(-1)^{2p-2r+2}f'(c_r)} = i^{2p-2r+2}\sqrt{f'(c_r)},\\
&\sqrt{(-1)^{p-r}P'(a_r)} = i^{p-r}\sqrt{P'(a_r)}, &&\sqrt{(-1)^{p-r+1}P(c_r)} = i^{p-r+1}\sqrt{P(c_r)},\\
&\sqrt{(-1)^{p-r+1}Q'(c_r)} = i^{p-r+1}\sqrt{Q'(c_r)}, &&\sqrt{(-1)^{p-r+1}Q(a_r)} = i^{p-r+1}\sqrt{Q(a_r)},\\
&\sqrt{(-1)^{p-r+1}F(a_r)} = (-i)^{p-r+1}\sqrt{F(a_r)}, &&\sqrt{(-1)^{p-r+1}F(c_r)} = (-i)^{p-r+1}\sqrt{F(c_r)},
\end{aligned}$$

where in each case, except the last two, the expression under the square root on the left side is merely an abbreviation for a product in which each factor is a *positive* difference, that is, for a difference in which the branch value subtracted has a lower position in the ascending order of the branch places than the branch value from which it is subtracted; for instance, in these equations, $(-1)^{p-r}P'(a_r)$ is an abbreviation for $(a_r - a_1)\ldots\ldots(a_r - a_{r-1})(a_{r+1} - a_r)\ldots\ldots(a_p - a_r)$. The same equations hold for the dissection (III). In the cases of the dissections (II), (IV), the i occurring on the right side of these equations is to be replaced by $-i$. The expressions $(-1)^{p-r+1}F(a_r)$, $(-1)^{p-r+1}F(c_r)$, in the last two of the equations above, are abbreviations for

$$(a_r - x_1)\ldots\ldots(a_r - x_{r-1})(x_r - a_r)\ldots\ldots(x_p - a_r),$$
$$(c_r - x_1)\ldots\ldots(c_r - x_{r-1})(x_r - c_r)\ldots\ldots(x_p - c_r),$$

respectively; they are introduced merely to enable us to write the formulæ more conveniently. In the standard case, when the branch values $c_1, a_1, \ldots$ are real quantities in ascending order of magnitude, the expressions occurring on the left sides in these equations, except the last two, are real quantities.

Consider now the functions

$$cl_r(u) = \frac{\vartheta(u \,|\, u^{a,\, c_r})}{\vartheta(u)} \Big/ \frac{\vartheta(0 \,|\, u^{a,\, c_r})}{\vartheta(0)}, \quad al_r(u) = \frac{\vartheta(u \,|\, u^{a,\, a_r})}{\vartheta(u)} \Big/ \frac{\vartheta'_r(0 \,|\, u^{a,\, a_r})}{\vartheta(0)},$$

where

$$\vartheta'_r(u \,|\, u^{a,\, a_r}) = \frac{\partial}{\partial V_r} \vartheta(u \,|\, u^{a,\, a_r}).$$

The quotient

$$cl_r(u) \Big/ \frac{\sqrt{x_1 - c_r} \ldots \sqrt{x_p - c_r}}{\sqrt{a_1 - c_r} \ldots \sqrt{a_p - c_r}}$$

is, on the dissected Riemann surface, a single-valued continuous function of each of $x_1, \ldots, x_p$, without zeros or infinities (B. 309); its value when the arguments u vanish, that is, when $x_1 = a_1, \ldots, x_p = a_p$, is $+1$. As the square of this function is $+1$, the function itself is $+1$. Thus, *in the case of the dissections* (I), (III), we have

$$cl_r(u) = \frac{(-i)^p \sqrt{F(c_r)}}{(-i)^{r-1}(i)^{p-r+1}\sqrt{P(c_r)}} = (-1)^{p-r+1} \frac{\sqrt{F(c_r)}}{\sqrt{P(c_r)}}$$

$$= \frac{\sqrt{(-1)^{p-r+1}F(c_r)}}{\sqrt{(-1)^{p-r+1}P(c_r)}}$$

(cf. Weierstrass, Math. Werke, I, 1894, p. 330, remembering that the branch places are here numbered in the reverse order. Cf. B. 569).

The function

$$al_r(u) = \frac{\vartheta(u \,|\, u^{a,\, a_r})}{\vartheta(u)} \Big/ \frac{\vartheta'_r(0 \,|\, u^{a,\, a_r})}{\vartheta(0)},$$

when $x_1, \ldots, x_p$ approach respectively to $a_1, \ldots, a_p$, is equal to t_r; hence the quotient

$$al_r(u) \Big/ \frac{\sqrt{x_1 - a_r} \ldots \sqrt{x_p - a_r}}{\sqrt{a_1 - a_r} \ldots \sqrt{a_p - a_r}},$$

wherein, in the denominator, there are $p - 1$ factors $\sqrt{a_i - a_r}$, is (B. 309) equal to $+1$, namely, for a dissection (I) or (III),

$$al_r(u) = \frac{(-i)^p \sqrt{F(a_r)}}{(-i)^{r-1}(i)^{p-r}\sqrt{P'(a_r)}} = (-1)^{p-r+1} i \frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}},$$

$$= \frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{\sqrt{(-1)^{p-r}P'(a_r)}}.$$

From these results we can deduce values for the quotient

$$\frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)}, \qquad \frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)};$$

we limit ourselves to the dissection (I).

We have
$$u_r^{c_1,\,a_1} + u_r^{c_2,\,a_2} + \ldots. + u_r^{c,\,a} = 0, \qquad (r = 1,\ 2, \ldots.., p)$$
and (B. 285)

$$\frac{\vartheta\left(u + \tfrac{1}{2}\Omega_q \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)}{\vartheta(u + \tfrac{1}{2}\Omega_q)} = e^{-\frac{1}{2}\pi i q' q} \frac{\vartheta(u)}{\vartheta\left(u \,\middle|\, \tfrac{1}{2}\binom{q'}{q}\right)};$$

if, therefore, in the expression obtained for $cl_r(u)$, we put $x_1, \ldots., x_p$ respectively at $c_1, \ldots., c_{r-1}, c, c_{r+1}, \ldots., c_p$, in which case $u_i = u_i^{a,\,c_r}$, we obtain

$$\frac{\vartheta^2(0)}{\vartheta^2(0\,|\,u^{a,\,c_r})} = (-1)^{p-r+1} i^p \frac{\sqrt{(c_1-c_r)(c_2-c_r)\ldots(c_{r-1}-c_r)(c-c_r)(c_{r+1}-c_r)\ldots(c_p-c_r)}}{\sqrt{P(c_r)}}$$

$$= (-1)^{p-r+1} i^p (-i)^{r-1} (i)^{p-r+1} \frac{\sqrt{Q'(c_r)}}{\sqrt{P(c_r)}}$$

$$= \frac{\sqrt{Q'(c_r)}}{\sqrt{P(c_r)}} = \frac{\sqrt{(-1)^{p-r+1}Q'(c_r)}}{\sqrt{(-1)^{p-r+1}P(c_r)}}$$

$$= \frac{\sqrt{f'(c_r)/4}}{P(c_r)} = (-1)^{p-r+1} \frac{\sqrt{(-1)^{2p-2r+2}f'(c_r)/4}}{P(c_r)};$$

from this equation we define one of the square roots of $\sqrt{f'(c_r)/4}$ by the equation

$$(\sqrt{f'(c_r)/4})^{\frac{1}{2}} = (-1)^{p-r+1}\sqrt{P(c_r)}\,\vartheta(0)/\vartheta(0\,|\,u^{a,\,c_r}),$$

and hence obtain

$$\frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)} = \frac{\sqrt{F(c_r)}}{(\sqrt{f'(c_r)/4})^{\frac{1}{2}}},$$

and we put similarly

$$(\sqrt{(-1)^{2p-2r+2}f'(c_r)/4})^{\frac{1}{2}} \quad \text{or} \quad (i^{2p-2r+2}\sqrt{f'(c_r)})^{\frac{1}{2}}$$

equal to
$$(-i)^{p-r+1}(\sqrt{f'(c_r)/4})^{\frac{1}{2}};$$

that is
$$\sqrt{(-1)^{p-r+1}P(c_r)}\,\vartheta(0)/\vartheta(0\,|\,u^{a,\,c_r})$$

and so obtain
$$\frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)}=\frac{\sqrt{(-1)^{p-r+1}F(c_r)}}{\left(\sqrt{(-1)^{2p-2r+2}f'(c_r)/4}\right)^{\frac{1}{2}}}.$$

Cf. Weierstrass, Math. Werke, I, 1894, p. 135.

Next, if in $al_r(u)$ we put $x_1, \ldots, x_p$ respectively equal to $a_1, \ldots, a_{r-1}$, a, $a_{r+1}, \ldots, a_p$, and therefore u equal to $u^{a,\,a_r}$, we obtain
$$al_r(u)=\frac{1}{i}\left[\frac{\vartheta(0)}{\vartheta_r'(0\,|\,u^{a,\,a_r})}\right]^2\frac{1}{(DV_r)_a\,t},$$
where $(DV_r)_a\,t$ is the value when $x=t^{-2}$, and t is small, of the integral
$$\frac{\sqrt{f'(a_r)}}{P'(a_r)}\int_a^x\frac{P(x)}{x-a_r}\,\frac{dx}{2y},$$
and is therefore equal to
$$-\frac{\sqrt{f'(a_r)/4}}{P'(a_r)}\,t;$$
hence we have
$$\begin{aligned}i\left[\frac{\vartheta(0)}{\vartheta_r'(0)\,|\,u^{a,\,a_r})}\right]^2&\frac{P'(a_r)}{\sqrt{f'(a_r)/4}}\cdot\frac{1}{t}\\&=\frac{1}{t}\,i(-1)^{p-r+1}\,i^p\,\frac{\sqrt{(a_1-a_r)\ldots\ldots(a_{r-1}-a_r)(a_{r+1}-a_r)\ldots\ldots(a_p-a_r)}}{\sqrt{P'(a_r)}}\\&=\frac{1}{t}\,i(-1)^{p-r+1}\,i^p\,(-i)^{r-1}\,(i)^{p-r}=\frac{1}{t}.\end{aligned}$$

We define now one square root of $i\sqrt{f'(a_r)/4}$ by the equation
$$\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}=i(-1)^{p-r}\sqrt{P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r}),$$
and hence obtain
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)}=\frac{\sqrt{F(a_r)}}{\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}};$$
further, we put
$$\left(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4}\right)^{\frac{1}{2}},\quad\text{or}\quad\left(i^{2p-2r+1}\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}$$

equal to $\qquad(-i)^{p-r}\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}$,

that is, to $\qquad i^{p-r+1}\sqrt{P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r})$,

or $\qquad i\sqrt{(-1)^{p-r}P'(a_r)}\,\vartheta(0)/\vartheta_r'(0\,|\,u^{a,\,a_r})$,

and so obtain
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)} = i\,\frac{\sqrt{(-1)^{p-r+1}F(a_r)}}{\left(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4}\right)^{\frac{1}{2}}}.$$

Cf. Weierstrass, Math. Werke, I, 1894, p. 135.

The results thus obtained may be summarized as follows:

Let
$$\lambda_r = \frac{(-1)^{p-r}\sqrt{P'(a_r)}}{\left(i\sqrt{f'(a_r)/4}\right)^{\frac{1}{2}}} = \frac{\sqrt{(-1)^{p-r}P'(a_r)}}{\left(\sqrt{(-1)^{2p-2r+1}f'(a_r)/4}\right)^{\frac{1}{2}}},$$

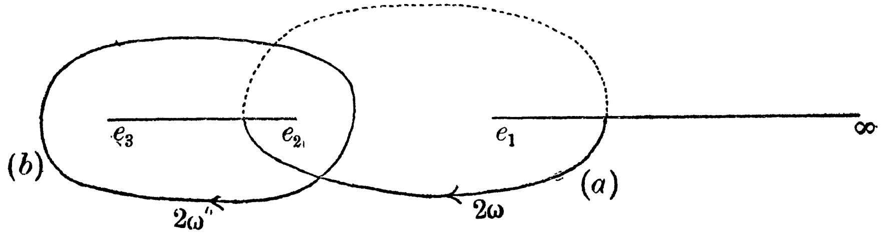
where the signification of the denominator is determined by
$$\frac{\vartheta_r'(0\,|\,u^{a,\,a_1})}{\vartheta(0)} = i\lambda_r;$$

also let
$$\mu_r = \frac{(-1)^{p-r+1}\sqrt{P(c_r)}}{\left(\sqrt{f'(c_r)/4}\right)^{\frac{1}{2}}} = \frac{\sqrt{(-1)^{p-r+1}P(c_r)}}{\left(\sqrt{(-1)^{2p-2r+2}f'(c_r)/4}\right)^{\frac{1}{2}}},$$

so that
$$\frac{\vartheta(0\,|\,u^{a,\,c_r})}{\vartheta(0)} = \mu_r;$$

then we have
$$\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(u)} = (-1)^{p-r}\lambda_r\frac{\sqrt{F(a_r)}}{\sqrt{P'(a_r)}},\quad \frac{\vartheta(u\,|\,u^{a,\,c_r})}{\vartheta(u)} = (-1)^{p-r+1}\mu_r\frac{\sqrt{F(c_r)}}{\sqrt{P(c_r)}}.$$

APPENDIX TO SECTION VI.—*The Elliptic case.*



Replacing c_1, a_1, c respectively by e_3, e_2, e_1 we have
$$u = u^{x,\,e_2} = u^{\infty,\,e_2} - u^{\infty,\,x} = \omega + \omega' - U, \text{ say,}$$

$$\begin{aligned}
\vartheta(u\,|\,u^{a,\,c_1}) &= \vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{1}{0}\right) = \quad \varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{0}{1}\right) = \quad \varepsilon\vartheta_0\left(\frac{U}{2\omega}\right),\\
\vartheta(u\,|\,u^{a,\,a_1}) &= \vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{1}{1}\right) = \quad \frac{1}{i}\,\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{0}{0}\right) = \frac{1}{i}\,\varepsilon\vartheta_3\left(\frac{U}{2\omega}\right),\\
\vartheta(u\,|\,u^{a,\,c}) &= \vartheta\left(\omega+\omega'-U\,\middle|\,\tfrac{1}{2}\binom{0}{1}\right) = -\frac{1}{i}\,\varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{1}{0}\right) = \quad i\varepsilon\vartheta_2\left(\frac{U}{2\omega}\right),\\
\vartheta(u) &= \vartheta\left(\omega+\omega'-U\right) \quad = \quad \varepsilon\vartheta\left(-U\,\middle|\,\tfrac{1}{2}\binom{1}{1}\right) = \quad \varepsilon\vartheta_1\left(\frac{U}{2\omega}\right),
\end{aligned}$$

where the notation is that of Weierstrass (cf. Formeln u. Lehrsätze, or Halphen, Fonct. Ellipt., I, p. 252), and ε is $e^{-(\eta+\eta')(U-\frac{1}{2}\omega-\frac{1}{2}\omega')}$; also, if $u/2\omega=\nu$,

$$\left[\frac{\partial}{\partial V}\vartheta\left(u\,|\,u^{a,\,a_1}\right)\right]_{u=0}=\frac{1}{2\omega\sqrt{(e_2-e_3)(e_2-e_1)}}\left[-\frac{\partial}{\partial\nu}\vartheta\left(u\,\middle|\,\tfrac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)\right]_{u=0}$$

$$=-\frac{\vartheta_1'}{2\omega\sqrt{(e_2-e_3)(e_2-e_1)}},$$

where ϑ_1' is the quantity so denoted by Halphen, p. 259.

Therefore our results give

$$\frac{-i\vartheta_1'/2\omega\sqrt{(e_2-e_3)(e_1-e_2)}}{\vartheta_3}=i\lambda_1=\frac{i}{\left(i\sqrt{(e_2-e_3)(e_2-e_1)}\right)^{\frac{1}{2}}}=\frac{i}{\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}},$$

$$\frac{\vartheta_2}{\vartheta_3}=\mu_1=-\frac{\sqrt{e_3-e_2}}{\left(\sqrt{(e_3-e_2)(e_3-e_1)}\right)^{\frac{1}{2}}}=\frac{\sqrt{e_2-e_3}}{\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}},$$

$$\frac{\vartheta_0}{\vartheta_3}=\mu_2=\frac{\sqrt{e_1-e_2}}{\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}}$$

and

$$\frac{1}{i}\,\frac{\vartheta_3(\nu)}{\vartheta_1(\nu)}=\lambda_1\sqrt{e_2-x}\quad=\frac{i}{\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}}\sqrt{x-e_2},$$

$$\frac{\vartheta_0(\nu)}{\vartheta_1(\nu)}=-\mu_1\frac{\sqrt{e_3-x}}{\sqrt{e_3-e_2}}=\mu_1\frac{\sqrt{x-e_3}}{\sqrt{e_2-e_3}}=\frac{1}{\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}}\sqrt{x-e_3},$$

$$i\,\frac{\vartheta_2(\nu)}{\vartheta_1(\nu)}=\mu_2\frac{\sqrt{e_1-x}}{\sqrt{e_1-e_2}}\quad=\frac{i}{\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}}\sqrt{x-e_1}.$$

Comparing the forms for ϑ_1'/ϑ_3, ϑ_2/ϑ_3, ϑ_0/ϑ_3 with those given by Halphen, Fonct. Elliptiq., I, p. 258, we immediately see that the determinations adopted by us for the square roots are given by

$$\left(\sqrt{(e_2-e_3)(e_1-e_2)}\right)^{\frac{1}{2}}=-\sqrt[4]{e_2-e_3}\sqrt[4]{e_1-e_2},$$

$$\left(\sqrt{(e_2-e_3)(e_1-e_3)}\right)^{\frac{1}{2}}=\sqrt[4]{e_2-e_3}\sqrt[4]{e_1-e_3},$$

$$\left(\sqrt{(e_1-e_2)(e_1-e_3)}\right)^{\frac{1}{2}}=\sqrt[4]{e_1-e_2}\sqrt[4]{e_1-e_3},$$

and that, with these determinations, the values found from our formulæ for the quotients $\vartheta_3(\nu)/\vartheta_1(\nu)$, $\vartheta_0(\nu)/\vartheta_1(\nu)$, $\vartheta_2(\nu)/\vartheta_1(\nu)$ agree with those given by Halphen, p. 260.

It is to be remarked that the formulæ of Halphen contain one, namely, $\sqrt[4]{e_1-e_3}=\sqrt{\frac{\pi}{2\omega}}\,\vartheta_3$, of which we have not put down the generalization. This is given by Thomae, Crelle, LXXI (1870), p. 216.

SECTION VII.

The expression of ϑ functions of two or more suffixes algebraically.

If in general the quotient $\vartheta(u\,|\,u^{a,\,b_1}+\ldots+u^{a,\,b_k})/\vartheta(u)$, where $b_1,\ldots,b_k$ are any finite branch places, and the characteristic is *not* reduced, be denoted by $q_{b_1b_2\ldots\ldots b_k}(u)$, and for $k>1$, the quotient, of two determinants of the p^{th} order,

$$\frac{\left|\dfrac{x_r^\lambda y_r}{\phi(x_r)},\ldots\ldots,\dfrac{y_r}{\phi(x_r)},\,x_r^\mu,\ldots\ldots,\,1\right|}{\left|x_r^{p-1}\quad,\ldots\ldots,\,x_r^\mu\qquad,\ldots\ldots,\,1\right|},$$

wherein $\lambda=\frac{1}{2}k-1$ or $\frac{1}{2}(k-3)$, $\mu=p-1-\frac{1}{2}k$ or $p-\frac{1}{2}(k+1)$, according as k is even or odd, and $\phi(x)=(x-b_1)\ldots\ldots(x-b_k)$ be denoted by $\pi_{b_1\ldots\,b_k}$; then (B. 312) we have

$$q_{b_1b_2}(u)=C_{q_{b_1}q_{b_2}}\pi_{b_1b_2},$$

namely,

$$\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,\,b_1})\,\vartheta(u\,|\,u^{a,\,b_2})}=C\,\frac{\left|\dfrac{y_r}{\phi(x_r)},\;x_r^{p-2},\ldots\ldots,\,x_r,\;1\right|}{\left|x_r^{p-1}\;,\,x_r^{p-2}\;,\ldots\ldots,\,x_r,\;1\right|},$$

$$=C\sum_{r=1}^{p}\frac{y_r}{(x_r-b_1)(x_r-b_2)\,F'(x_r)},$$

where C is a constant. We proceed to determine the value of C^2.

Let $u^{a,\,b_2}=\frac{1}{2}\binom{q'}{q}$, $u^{a,\,b_1}=\frac{1}{2}\binom{k'}{k}$, the notation being as previously explained. Then if $v=u+u^{a,\,b_1}+\Omega_{m,\,m'}$, where $\Omega_{m,\,m'}$ denotes a period, we have (B. 286)

$$\frac{\vartheta(v\,|\,u^{a,\,b_2})}{\vartheta(v)}=e^{\pi i\,(mq'-m'q)}\,\frac{\vartheta(u+u^{a,\,b_1}\,|\,u^{a,\,b_2})}{\vartheta(u+u^{a,\,b_1})}$$

$$=e^{\pi i\,(mq'-m'q)}e^{-\frac{1}{2}\pi ik'q}\,\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})}{\vartheta(u\,|\,u^{a,\,b_1})},$$

where the characteristic denoted by $u^{a,\,b_1} + u^{a,\,b_2}$ is unreduced, namely, is $\frac{1}{2}\begin{pmatrix} q' + k' \\ q + k \end{pmatrix}$; therefore

$$\frac{\vartheta(u\,|\,u^{a,\,b_1} + u^{a,\,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,\,b_1})\,\vartheta(u\,|\,u^{a,\,b_2})} = e^{\pi i(mq' - m'q)} e^{\frac{1}{2}\pi i k' q}\,\frac{\vartheta(v\,|\,u^{a,\,b_2})}{\vartheta(v)} \Big/ \frac{\vartheta(u\,|\,u^{a,\,b_2})}{\vartheta(u)};$$

and if

$$\begin{aligned} v &= u^{z_1,\,a_1} + \ldots\ldots + u^{z_p,\,a_p}, \\ u &= u^{x_1,\,a_1} + \ldots\ldots + u^{x_p,\,a_p}, \\ F(x) &= (x - x_1)\ldots\ldots(x - x_p), \\ G(x) &= (x - z_1)\ldots\ldots(x - z_p), \end{aligned}$$

the right side is equal (Section VI) to

$$e^{\pi i(mq' - m'q)} e^{\frac{1}{2}\pi i k' q}\,\frac{\sqrt{G(b_2)}}{\sqrt{F(b_2)}}.$$

Now from $\qquad u^{z_1,\,x_1} + \ldots\ldots + u^{z_p,\,x_p} + u^{b_1,\,a} = \Omega_{m,\,m'}$,

it follows that $z_1, \ldots\ldots, z_p, b_1$ are the zeros, and $x_1, \ldots\ldots, x_p, a$ are the poles of a rational function, therefore given by

$$\frac{y}{F(x)} + (x - b_1)\sum_{i=1}^{p} \frac{y_i}{x_i - b_1}\,\frac{1}{(x - x_i)\,F'(x_i)}$$

(cf. B. 318), and hence it follows that

$$\frac{G(b_2)}{F(b_2)} = (b_1 - b_2)\Big[\tfrac{1}{2}\sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)(x_i - b_2)\,F'(x_i)}\Big]^2,$$

and therefore that

$$\frac{\vartheta^2(u\,|\,u^{a,\,b_1} + u^{a,\,b_2})\,\vartheta^2(u)}{\vartheta^2(u\,|\,u^{a,\,b_1})\,\vartheta^2(u\,|\,u^{a,\,b_2})} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(b_1 - b_2)\Big[\tfrac{1}{2}\sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)(x_i - b_2)\,F'(x_i)}\Big]^2,$$

where for $B_1 = \frac{1}{2}\begin{pmatrix} k' \\ k \end{pmatrix}$, $B_2 = \frac{1}{2}\begin{pmatrix} q' \\ q \end{pmatrix}$ we have $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = e^{\pi i k' q}$.

This equation is true in the case of either of the dissections (I), (II), (III), (IV); supposing the dissection to be of the kind (I), and that, in the ascending order of the branch places, b_2 is higher than b_1, we have $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(b_1 - b_2) = b_2 - b_1$. Hence, retaining the notation $u^{a,\,b_1} + u^{a,\,b_2}$ for the sum, *without reduction*, of the characteristics denoted by $u^{a,\,b_1}$ and $u^{a,\,b_2}$, we may write

$$\frac{\vartheta(u\,|\,u^{a,\,b_1} + u^{a,\,b_2})\,\vartheta(u)}{\vartheta(u\,|\,u^{a,\,b_1})\,\vartheta(u\,|\,u^{a,\,b_2})} = \varepsilon_{b_1 b_2}\sqrt{b_2 - b_1}\,\tfrac{1}{2}\sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)(x_i - b_2)\,F'(x_i)},$$

where ε_{b_1, b_2} is ± 1, but is perfectly definite. We do not here determine its value.

Now let b_1, b_2, b_3 be any three finite branch places, which, for definiteness, we suppose to be in ascending order. Then we can determine in a similar way the constant factor in the algebraic expression of the square of the function

$$\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2}+u^{a,\,b_3})\,\vartheta^2(u)}{\vartheta(u\,|\,u^{a,\,b_1})\,\vartheta(u\,|\,u^{a,\,b_2})\,\vartheta(u\,|\,u^{a,\,b_3})}\,.$$

For denoting $\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2}+u^{a,\,b_3})$ momentarily by $\vartheta_{123}(u)$, etc., and putting

$$v = u^{z_1,\,a_1} + \ldots\ldots + u^{z_p,\,a_p},$$
$$u \equiv v + u^{a,\,b_1},$$

where the sign $\equiv$ indicates that the expressions on the two sides may differ by a period, the function

$$\frac{\vartheta_{123}^2(u)\,\vartheta_1^2(u)}{\vartheta_{12}^2(u)\,\vartheta_{13}^2(u)} = \frac{\vartheta_{23}^2(v)\,\vartheta^2(v)}{\vartheta_2^2(v)\,\vartheta_3^2(v)}$$
$$= \left(\frac{B_2}{B_3}\right)(b_2-b_3)\left[\tfrac{1}{2}\sum_{i=1}^{p}\frac{s_i}{(z_i-b_2)(z_i-b_3)\,G'(z_i)}\right]^2,$$

where s_i is the value of y when x is at z_i, and $z_1, \ldots\ldots, z_p$ are zeros of the rational function

$$\frac{y}{F(x)} + (x-b_1)\sum_{i=1}^{p}\frac{y_i/(x_i-b_i)}{(x-x_i)\,F'(x_i)}\,;$$

expressing that $z_1, \ldots\ldots, z_p$ are zeros of this function we derive that

$$\sum_{r=1}^{p}\frac{s_r}{(z_r-b_2)(z_r-b_3)\,G'(z_r)}$$
$$= -\sum_{r=1}^{p}\frac{(z_r-b_1)\,F(z_r)}{(z_r-b_2)(z_r-b_3)\,G'(z_r)}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(z_r-x_i)\,F'(x_i)}$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\sum_{r=1}^{p}\frac{(z_r-b_1)\,F(z_r)\,y_i}{(x_i-b_1)(z_r-x_i)\,F'(x_i)\,G'(z_r)}\left(\frac{1}{z_r-b_2}-\frac{1}{z_r-b_3}\right)$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)}\sum_{r=1}^{p}\frac{(z_r-b_1)(x_i-b_2)\,F(z_r)}{(z_r-x_i)(z_r-b_2)\,G'(z_r)}$$
$$+\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_3)\,F'(x_i)}\sum_{r=1}^{p}\frac{(z_r-b_1)(x_i-b_3)\,F(z_r)}{(z_r-x_i)(z_r-b_3)\,G'(z_r)}$$
$$= -\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)}\left[x_i-b_2+(b_2-b_1)\,\frac{F(b_2)}{G(b_2)}\right]$$
$$+\frac{1}{b_2-b_3}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_3)\,F'(x_i)}\left[x_i-b_3+(b_3-b_1)\,\frac{F(b_3)}{G(b_3)}\right]$$
$$= \frac{b_2-b_1}{b_3-b_2}\,\frac{F(b_2)}{G(b_2)}\,\pi_{b_1 b_2} - \frac{b_3-b_1}{b_3-b_2}\,\frac{F(b_3)}{G(b_3)}\,\pi_{b_1 b_3}\,;$$

now we have seen that

$$4\,\frac{G(b_2)}{F(b_2)} = (b_1 - b_2)\,\pi^2_{b_1, b_3}, \quad 4\,\frac{G(b_3)}{F(b_3)} = (b_1 - b_3)\,\pi^2_{b_1, b_2};$$

therefore

$$\tfrac{1}{2}\sum_{r=1}^{p} \frac{s_r}{(z_r - b_2)(z_r - b_3)\,G'(z_r)} = -\,\frac{2}{b_3 - b_2}\left(\frac{1}{\pi_{b_1 b_2}} - \frac{1}{\pi_{b_1 b_3}}\right)$$

$$= +\,\frac{2}{(b_3 - b_2)\,\pi_{b_1 b_2}\,\pi_{b_1 b_3}} \sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)\,F'(x_i)} \left(\frac{1}{x_i - b_2} - \frac{1}{x_i - b_3}\right)$$

$$= -\,\frac{2}{\pi_{b_1 b_2}\,\pi_{b_1 b_3}} \sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)(x_i - b_2)(x_i - b_3)\,F'(x_i)}$$

$$= -\,\frac{2\pi_{b_1 b_2 b_3}}{\pi_{b_1 b_2}\,\pi_{b_1 b_3}}\,.$$

Hence we have, for either one of the dissections (I), (II), (III), (IV),

$$\frac{\vartheta^2(u \,|\, u^{a,\,b_1} + u^{a,\,b_2} + u^{a,\,b_3})\,\vartheta^4(u)}{\vartheta^2(u \,|\, u^{a,\,b_1})\,\vartheta^2(u \,|\, u^{a,\,b_2})\,\vartheta^2(u \,|\, u^{a,\,b_3})} = \frac{\vartheta^2_{123}(u)\,\vartheta^4(u)}{\vartheta^2_1(u)\,\vartheta^2_2(u)\,\vartheta^2_3(u)}$$

$$= \frac{\vartheta^2_{123}(u)\,\vartheta^2_1(u)}{\vartheta^2_{12}(u)\,\vartheta^2_{13}(u)}\;\frac{\vartheta^2_{12}(u)\,\vartheta^2(u)}{\vartheta^2_1(u)\,\vartheta^2_2(u)}\cdot\frac{\vartheta^2_{13}(u)\,\vartheta^2(u)}{\vartheta^2_1(u)\,\vartheta^2_3(u)}$$

$$= \left(\frac{B_2}{B_3}\right)(b_2 - b_3)\left(\frac{B_1}{B_2}\right)(b_1 - b_2)\left(\frac{B_1}{B_3}\right)(b_1 - b_3)\,4\,\frac{\pi^2_{b_1 b_2 b_3}}{\pi^2_{b_1 b_2}\,\pi^2_{b_1 b_3}}\cdot\tfrac{1}{4}\,\pi^2_{b_1 b_2}\cdot\tfrac{1}{4}\,\pi^2_{b_1 b_2},$$

which, in the case of the dissection (I), is equal to

$$\tfrac{1}{4}\,(b_3 - b_2)(b_2 - b_1)(b_3 - b_1)\,\pi^2_{b_1 b_2 b_3};$$

we can therefore write, on the whole,

$$\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)}$$

$$= \varepsilon_{123}\sqrt{(b_3 - b_2)(b_2 - b_1)(b_3 - b_1)}\;\tfrac{1}{2}\sum_{i=1}^{p} \frac{y_i}{(x_i - b_1)(x_i - b_2)(x_i - b_3)\,F'(x_i)},$$

wherein the ± sign denoted by ε_{123} is perfectly definite, it being remembered that in $\vartheta_{123}(u)$ the characteristic is the sum, *without reduction*, of the characteristics in $\vartheta_1(u)$, $\vartheta_2(u)$, $\vartheta_3(u)$.

We now proceed to show that

$$\varepsilon_{123} = -\,\varepsilon_{23}\,\varepsilon_{31}\,\varepsilon_{12},$$

where $\varepsilon_{12} = \varepsilon_{21}$, $\varepsilon_{31} = \varepsilon_{13}$, $\varepsilon_{32} = \varepsilon_{23}$ are the signs before introduced.

For denoting $\pi_{b_1 b_2}$ as $\pi_{12}(u)$, and remembering that it is a periodic function, we have found that

$$\pi_{23}(v) = \pi_{23}(u + u^{a,\, b_1}) = -4\, \frac{\pi_{123}(u)}{\pi_{12}(u)\, \pi_{13}(u)},$$

while also, in case of the dissection (I),

$$\begin{aligned}\pi_{23}(v) &= \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\, \frac{\vartheta_{23}(v)\, \vartheta(v)}{\vartheta_2(v)\, \vartheta_3(v)} = \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\, \frac{\vartheta_{123}(u)\, \vartheta_1(u)}{\vartheta_{12}(u)\, \vartheta_{13}(u)} \\ &= \frac{2\varepsilon_{23}}{\sqrt{b_3 - b_2}}\, \frac{\frac{1}{2}\varepsilon_{123}\, \pi_{123}(u) \sqrt{(b_3 - b_2)(b_2 - b_1)(b_3 - b_1)}}{\frac{1}{2}\varepsilon_{12}\, \pi_{12}(u) \sqrt{b_2 - b_1}\, \frac{1}{2}\varepsilon_{13}\, \pi_{13}(u) \,.\, \sqrt{b_3 - b_1}} \\ &= \frac{4\varepsilon_{23}\, \varepsilon_{123}}{\varepsilon_{12}\, \varepsilon_{13}}\, \frac{\pi_{123}(u)}{\pi_{12}(u)\, \pi_{13}(u)},\end{aligned}$$

and comparing this with the former result we have the equation in question.

The obvious equations

$$\pi_{13} - \pi_{23} = (b_1 - b_2)\, \pi_{123},$$

$$(b_2 - b_3)\, \pi_{23} + (b_3 - b_1)\, \pi_{31} + (b_1 - b_2)\, \pi_{12} = 0,$$

give

$$-(b_2 - b_3)(b_3 - b_1)(b_1 - b_2)\, \pi_{123}^2 = (b_2 - b_3)\, \pi_{23}^2 + (b_3 - b_1)\, \pi_{31}^2 + (b_1 - b_2)\, \pi_{12}^2,$$

and therefore, for the dissections (I), (II), (III), (IV),

$$\begin{aligned}&-\binom{B_2}{B_3}\binom{B_3}{B_1}\binom{B_1}{B_2}\, \vartheta^2(u)\, \vartheta_{123}^2(u) \\ &\qquad = \binom{B_2}{B_3}\, \vartheta_1^2(u)\, \vartheta_{23}^2(u) + \binom{B_3}{B_1}\, \vartheta_{31}^2(u)\, \vartheta_2^2(u) + \binom{B_1}{B_2}\, \vartheta_{12}^2(u)\, \vartheta_3^2(u);\end{aligned}$$

in particular, for the dissection (I) we have

$$\vartheta^2(u)\, \vartheta_{123}^2(u) = \vartheta_1^2(u)\, \vartheta_{23}^2(u) - \vartheta_2^2(u)\, \vartheta_{31}^2(u) + \vartheta_3^2(u)\, \vartheta_{12}^2(u),$$

an equation which, for our purpose, is of importance.

The corresponding expressions for functions of four suffixes may be set down here. We have

$$\pi_{1234} = \sum_{r=1}^{p} \sum_{s=1}^{p} \frac{y_r\, y_s}{\phi(x_r)\, \phi(x_s)}\, \frac{(x_r - x_s)^2}{F'(x_r)\, F'(x_s)},$$

and hence we find

$$\pi_{12}\, \pi_{34} - \pi_{13}\, \pi_{24} = (b_2 - b_3)(b_1 - b_4)\, \pi_{1234},$$

and

$$(b_2 - b_3)(b_1 - b_4)\, \pi_{23}\, \pi_{14} + (b_3 - b_1)(b_2 - b_4)\, \pi_{31}\, \pi_{24} + (b_1 - b_2)(b_3 - b_4)\, \pi_{12}\, \pi_{34} = 0;$$

thence we deduce

$$(b_2-b_3)(b_3-b_1)(b_1-b_2)(b_1-b_4)(b_2-b_4)(b_3-b_4)\,\pi^2_{1234}$$
$$=(b_2-b_3)(b_1-b_4)\,\pi^2_{23}\pi^2_{14}+(b_3-b_1)(b_2-b_4)\,\pi^2_{31}\pi^2_{24}+(b_1-b_2)(b_3-b_4)\,\pi^2_{12}\pi^2_{34}.$$

SECTION VIII.

The first terms in the expansion of a function of one, two or three suffixes.

Putting

$$\vartheta(u\,|\,u^{a,\,a_r})=\vartheta(0)[A_ru_1+\ldots\ldots+P_ru_p+(u_1,\ldots\ldots,u_p)_2+(u_1,\ldots\ldots,u_p)_3+\ldots\ldots],$$

where $(u_1,\ldots\ldots,u_p)_k$ denotes an integral polynomial in $u_1,\ldots\ldots,u_p$ of order k, we have from the results of Section VI, taking $x_1,\ldots\ldots,x_p$ near to $a_1,\ldots\ldots,a_p$ respectively, and $x_k=a_k+t_k^2$, the equation

$$\begin{aligned}A_ru_1+\ldots\ldots+P_ru_p&=\frac{(-1)^{p-r}\lambda_r i^p t_r}{(-1)^{p-r}\,i^{p-1}}=i\lambda_r t_r\\&=i\lambda_r V_r\\&=\frac{1}{\lambda_r}\,[u_p+u_{p-1}\chi_1(a_r)+\ldots\ldots+u_1\chi_{p-1}(a_r)]\,;\end{aligned}$$

hence the first terms in the expansion of

$$\lambda_r\,\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(0)},$$

that is, the linear terms, are

$$u_p+u_{p-1}\chi_1(a_r)+\ldots\ldots+u_1\chi_{p-1}(a_r)\,;$$

this may be symbolically denoted by

$$\frac{P(\xi)}{\xi-a_r},$$

where, after the division of $P(\xi)$ by $\xi-a_r$, we are to replace ξ^{i-1} by u_i and $\xi^{(0)}$ by u_1.

Consider next

$$\frac{\vartheta(u\,|\,u^{a,\,b_1}+u^{a,\,b_2})}{\vartheta(u)}$$
$$=\varepsilon_{b_1b_2}\sqrt{b_2-b_1}\,\frac{\vartheta(u|u^{a,\,b_1})}{\vartheta(u)}\cdot\frac{\vartheta(u\,|\,u^{a,\,b_2})}{\vartheta(u)}\cdot\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)\,(x_i-b_2)\,F'(x_i)},$$

and suppose that b_1, b_2 are taken from among the p branch places $a_1, \ldots, a_p$ and are in ascending order, it will be sufficient to consider the case when they are respectively a_1 and a_2; then denoting $\varepsilon_{a_1 a_2}$ by ε_{12} and supposing $x_k = a_k + t_k^2$, for $k = 1, \ldots, p$, the quantities $t_1, \ldots, t_p$ being small, we have to the first approximation

$$\begin{aligned}\frac{\vartheta(u \mid u^{a, a_1} + u^{a, a_2})}{\vartheta(u)} &= \tfrac{1}{2}\varepsilon_{12}\sqrt{a_2 - a_1}\,.\,i\lambda_1 t_1\,.\,i\lambda_2 t_2\left(\frac{\sqrt{f'(a_1)}}{P'(a_1)}\cdot\frac{1}{t_1} - \frac{\sqrt{f'(a_2)}}{P'(a_2)}\cdot\frac{1}{t_2}\right)\frac{1}{a_1 - a_2} \\ &= \tfrac{1}{2}\varepsilon_{12}\sqrt{a_2 - a_1}\,.\,i^2\lambda_1\lambda_2\left(\frac{2}{i\lambda_1^2}t_2 - \frac{2}{i\lambda_2^2}t_1\right)\frac{1}{a_1 - a_2} \\ &= i\varepsilon_{12}\sqrt{a_2 - a_1}\,\frac{1}{\lambda_1\lambda_2}(\lambda_2^2 V_2 - \lambda_1^2 V_1)\frac{1}{a_1 - a_2} \\ &= -\varepsilon_{12}\sqrt{a_2 - a_1}\,\frac{1}{\lambda_1\lambda_2}\left[u_{p-1}\frac{\chi_1(a_1) - \chi_1(a_2)}{a_1 - a_2}\right. \\ &\qquad\qquad\left. + \ldots + u_1\frac{\chi_{p-1}(a_1) - \chi_{p-1}(a_2)}{a_1 - a_2}\right],\end{aligned}$$

so that if

$$\psi_{p-i}(a_1, a_2) = \frac{\chi_{p-i}(a_1) - \chi_{p-i}(a_2)}{a_1 - a_2},$$

the expansion of the function

$$-\frac{\varepsilon_{12}\lambda_1\lambda_2}{\sqrt{a_2 - a_1}}\,\frac{\vartheta(u \mid u^{a, a_1} + u^{a, a_2})}{\vartheta(0)}$$

begins with

$$u_{p-1} + u_{p-2}\psi_2(a_1, a_2) + \ldots + u_1\psi_{p-1}(a_1, a_2);$$

this expression may symbolically, as just explained, be denoted by

$$\left[\frac{P(\xi)}{\xi - a_1} - \frac{P(\xi)}{\xi - a_2}\right]\frac{1}{a_1 - a_2},$$

that is, by

$$P(\xi)/(\xi - a_1)(\xi - a_2),$$

where, after division, we are to replace ξ^{i-1} by u_i and $\xi^{(0)}$ by u_1. It is to be borne in mind that in this statement the characteristic denoted by $u^{a, a_1} + u^{a, a_2}$ is *unreduced.*

Putting now

$$\zeta_{23} = \varepsilon_{23}\sqrt{a_3 - a_2},\ \zeta_{31} = \varepsilon_{31}\sqrt{a_3 - a_1},\ \zeta_{12} = \varepsilon_{12}\sqrt{a_2 - a_1},$$

where a_1, a_2, a_3 are in ascending order and chosen from among the p branch places $a_1, \dots, a_p$, and $\varepsilon_{23} = \varepsilon_{a_2 a_3}$, etc., as before, consider the equation

$$\frac{\vartheta(u\,|\,u^{a,\,a_1} + u^{a,\,a_2} + u^{a,\,a_3})}{\vartheta(u)}$$
$$= -\tfrac{1}{2}\zeta_{23}\zeta_{31}\zeta_{12}\,\frac{\vartheta(u\,|\,u^{a,\,a_1})}{\vartheta(u)}\cdot\frac{\vartheta(u\,|\,u^{a,\,a_2})}{\vartheta(u)}\cdot\frac{\vartheta(u\,|\,u^{a,\,a_3})}{\vartheta(u)}\sum_{i=1}^{p}\frac{y_i/P'(x_i)}{(x_i-a_1)(x_i-a_2)(x_i-a_3)},$$

which has already been established (Section VII). Writing for the moment

$$\frac{\vartheta(u\,|\,u^{a,\,a_1})}{\vartheta(u)} = q_1, \quad \frac{\vartheta(u\,|\,u^{a,\,a_2})}{\vartheta(u)} = q_2, \text{ etc.,} \quad \frac{\vartheta(u\,|\,u^{a,\,a_1} + u^{a,\,a_2})}{\vartheta(u)} = q_{12},$$

we have

$$q_1 q_2 q_3 \sum_{i=1}^{p}\frac{y_i}{(x_i-a_1)(x_i-a_2)(x_i-a_3)\,P'(x_i)}$$
$$= \frac{q_1 q_2 q_3}{a_2-a_3}\left[\sum_{i=1}^{p}\frac{y_i}{(x_i-a_1)(x_i-a_2)\,P'(x_i)} - \sum_{i=1}^{p}\frac{y_i}{(x_i-a_1)(x_i-a_3)\,P'(x_i)}\right],$$
$$= 2\,\frac{q_1 q_2 q_3}{a_2-a_3}\left[\frac{q_{12}}{q_1 q_2}\cdot\frac{1}{\zeta_{12}} - \frac{q_{13}}{q_1 q_3}\,\frac{1}{\zeta_{13}}\right],$$
$$= \frac{2}{a_2-a_3}\left[\frac{q_3 q_{12}}{\zeta_{12}} - \frac{q_2 q_{13}}{\zeta_{13}}\right],$$

and, for small values of the arguments the first terms of this expression, as follows by what has just previously been established, are those symbolically denoted by

$$-\frac{2}{a_2-a_3}\,\frac{1}{\lambda_1\lambda_2\lambda_3}\left[\frac{P(\xi)}{\xi-a_3}\cdot\frac{P(\xi')}{(\xi'-a_1)(\xi'-a_2)} - \frac{P(\xi')}{\xi'-a_2}\,\frac{P(\xi)}{(\xi-a_1)(\xi-a_3)}\right],$$

where ξ' is a symbol equivalent, in its interpretation, with ξ; hence, if we put

$$\phi(x) = \frac{P(x)}{(x-a_1)(x-a_2)(x-a_3)},$$

the first terms in $\vartheta(u\,|\,u^{a,\,a_1} + u^{a,\,a_2} + u^{a,\,a_3})/\vartheta(u)$ are those symbolically denoted by

$$\frac{\zeta_{23}\zeta_{31}\zeta_{12}}{\lambda_1\lambda_2\lambda_3}\,\phi(\xi)\,\phi(\xi')\left[(\xi'-a_3)(\xi-a_1)(\xi-a_2) - (\xi-a_2)(\xi'-a_1)(\xi'-a_3)\right]/(a_2-a_3);$$

now, so far as the interpretation is concerned,

$$
\begin{aligned}
&(\xi'-a_3)(\xi-a_1)(\xi-a_2)-(\xi-a_2)(\xi'-a_1)(\xi'-a_3)\\
=\;&(\xi-a_3)(\xi'-a_1)(\xi'-a_2)-(\xi'-a_2)(\xi-a_1)(\xi-a_3)\\
=\;&\tfrac{1}{2}\left[(\xi-a_1)\{(\xi'-a_3)(\xi-a_2)-(\xi'-a_2)(\xi-a_3)\}\right.\\
&\qquad\qquad\left.-(\xi'-a_1)\{(\xi-a_2)(\xi'-a_3)-(\xi-a_3)(\xi'-a_2)\}\right]\\
=\;&\tfrac{1}{2}\left[(\xi-a_1)(\xi-\xi')(a_2-a_3)-(\xi'-a_1)(\xi-\xi')(a_2-a_3)\right]\\
=\;&\tfrac{1}{2}(\xi-\xi')^2(a_2-a_3).
\end{aligned}
$$

Thus, finally, we may state the result by saying that the function

$$+\frac{\lambda_1\lambda_2\lambda_3}{\zeta_{23}\zeta_{31}\zeta_{12}}\,\frac{\vartheta(u\,|\,u^{a,\,a_1}+u^{a,\,a_2}+u^{a,\,a_3})}{\vartheta(0)}$$

has for the lowest terms in its expansion those of the second degree in $u_1, \ldots, u_p$, which are symbolically denoted by

$$+\tfrac{1}{2}\phi(\xi)\,\phi(\xi')(\xi-\xi')^2.$$

Thus, for example, when $p=3$, $\phi(\xi)=1$, and the function begins with the terms denoted by $+\frac{1}{2}(\xi-\xi')^2$, namely,

$$+\tfrac{1}{2}(u_3u_1-2u_2^2+u_1u_3)=u_1u_3-u_2^2;$$

while, for $p=4$, $\phi(\xi)=\xi-a_4$, and the function begins with

$$
\begin{aligned}
&+\tfrac{1}{2}\left[\xi\xi'-(\xi+\xi')\,a_4+a_4^2\right]\left[\xi^2-2\xi\xi'+\xi'^2\right],\\
=\;&-\xi^2\xi'^2+\tfrac{1}{2}\xi^3\xi'+\tfrac{1}{2}\xi\xi'^3\\
&-\tfrac{1}{2}a_4(\xi^3+\xi'^3-\xi\xi'^2-\xi'\xi^2)\\
&-a_4^2(\xi\xi'-\tfrac{1}{2}\xi^2-\tfrac{1}{2}\xi'^2)\\
=\;&-u_3^2+u_2u_4+a_4(u_2u_3-u_1u_4)+a_4^2(u_1u_3-u_2^2).
\end{aligned}
$$

Cf. Schottky, "Abriss einer Theorie der Abelschen Functionen von drei Variabeln" (Leipzig, 1880), p. 149, and Burkhardt, Math. Annal., XXXII (1888), p. 442. The form $\phi(x)=P(x)/(x-a_1)\ldots\ldots(x-a_k)$ is in fact that denoted, by Klein, by $\phi^{p+1-2\mu}$, associated with $\vartheta(u\,|\,u^{a,\,a_1}+\ldots\ldots+u^{a,\,a_k})$.

The results obtained in this section may be summarized as follows: Let

$$\sigma_r(u)=\lambda_r\,\frac{\vartheta(u\,|\,u^{a,\,a_r})}{\vartheta(0)},$$

$$\sigma_{r,\,s}(u)=-\frac{\lambda_r\lambda_s}{\zeta_{rs}}\,\frac{\vartheta(u\,|\,u^{a,\,a_r}+u^{a,\,a_s})}{\vartheta(0)},$$

$$\sigma_{r,\,s,\,t}(u)=\frac{\lambda_r\lambda_s\lambda_t}{\zeta_{rs}\zeta_{rt}\zeta_{st}}\,\frac{\vartheta(u\,|\,u^{a,\,a_r}+u^{a,\,a_s}+u^{a,\,a_t})}{\vartheta(0)},$$

where the characteristics are *unreduced;* then the first terms in the expansions of these functions are, respectively, those denoted by

$$P(\xi)/(\xi-a_r),\quad P(\xi)/(\xi-a_r)(\xi-a_s),\quad \tfrac{1}{2}\phi(\xi)\phi(\xi')(\xi-\xi')^2,$$

where

$$\phi(\xi)=P(\xi)/(\xi-a_r)(\xi-a_s)(\xi-a_t),\quad \xi^{i-1}=u_i,\quad \xi^{(0)}=u_1.$$

SECTION IX.

Expression of the function $\vartheta(u|u^{a,b_1}+\dots\dots+u^{a,b_k})$ *in terms of functions* $\vartheta(u)$, $\vartheta(u|u^{a,b_r})$, $\vartheta(u|u^{a,b_r}+u^{a,b_s})$.

We have, for the dissection (I), and when the characteristic denoted by $u^{a,b_1}+u^{a,b_2}$ is unreduced, and the places $b_1, b_2, b_3, \dots\dots$ are in ascending order,

$$\frac{\vartheta(u|u^{a,b_1}+u^{a,b_2})\,\vartheta(u)}{\vartheta(u|u^{a,b_1})\,\vartheta(u|u^{a,b_2})}=\epsilon_{12}\sqrt{b_2-b_1}\,\tfrac{1}{2}\pi_{12}=\zeta_{12}\,\tfrac{1}{2}\sum_{i=1}^{p}\frac{y_i}{(x_i-b_1)(x_i-b_2)\,F'(x_i)};$$

we write

$$\phi_{12}(u)=\frac{\vartheta(u|u^{a,b_1}+u^{a,b_2})}{\zeta_{12}},$$

the characteristic being unreduced. Further we write, for the present,

$$\vartheta_1(u)=\vartheta(u|u^{a,b_1}),\quad \vartheta_{12}=\vartheta(u|u^{a,b_1}+u^{a,b_2}),\ \text{etc.}$$

Then we have found

$$\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)}=-\zeta_{23}\,\zeta_{13}\,\zeta_{12}\,\tfrac{1}{2}\pi_{123},$$

and

$$\pi_{13}-\pi_{23}=(b_1-b_2)\,\pi_{123},$$

so that

$$\frac{1}{\zeta_{13}}\,\frac{\vartheta_{13}(u)\,\vartheta(u)}{\vartheta_1(u)\,\vartheta_3(u)}-\frac{1}{\zeta_{23}}\cdot\frac{\vartheta_{23}(u)\,\vartheta(u)}{\vartheta_2(u)\,\vartheta_3(u)}=-\frac{b_1-b_2}{\zeta_{23}\,\zeta_{31}\,\zeta_{12}}\,\frac{\vartheta_{123}(u)\,\vartheta^2(u)}{\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)},$$

or

$$\begin{aligned}\vartheta(u)\,\vartheta_{123}(u)&=-\frac{\zeta_{23}\,\zeta_{31}\,\zeta_{12}}{b_1-b_2}\left[\vartheta_2(u)\,\phi_{13}(u)-\vartheta_1(u)\,\phi_{23}(u)\right],\\&=\zeta_{23}\,\zeta_{31}\,\zeta_{12}\begin{vmatrix}\vartheta_1(u)&\vartheta_2(u)\\\phi_{13}(u)&\phi_{23}(u)\end{vmatrix}\div(b_1-b_2),\end{aligned}$$

and it is manifest that the function

$$\frac{\vartheta_1(u)\,\phi_{23}(u)-\vartheta_2(u)\,\phi_{13}(u)}{b_1-b_2},$$

being equal to $\vartheta(u)\,\vartheta_{123}(u)/\zeta_{23}\zeta_{31}\zeta_{12}$, is unaltered by the interchange of any two of the suffixes 1, 2, 3. Further (B. 285), if $a^{a,\,b_3}=\binom{p'}{p}$, $u^{a,\,b_1}+u^{a,\,b_2}=\binom{q'}{q}$,

$$\begin{aligned}\phi_{12}(u+u^{a,\,b_3}) &= e^{\lambda_p(u)-2\pi i p'_q}\,\frac{\vartheta_{123}(u)}{\zeta_{12}}\\ &= e^{\lambda_p(u)-2\pi i p'_q}\,\zeta_{13}\,\zeta_{23g}\,\frac{\vartheta_1(u)\,\phi_{23}(u)-\vartheta_2(u)\,\phi_{13}(u)}{\vartheta(u)\,.\,(b_1-b_2)};\end{aligned}$$

we make frequent use of this fact in what immediately follows; for the sake of brevity the right hand will be denoted, more shortly, by

$$e^{\lambda_p(u)-2\pi i p'_q}\,\zeta_{13}\,\zeta_{23}\,\frac{\vartheta_1\,\phi_{23}-\vartheta_2\,\phi_{13}}{\vartheta\,.\,(b_1-b_2)}\,.$$

Also we put

$$\nabla_{12\,\ldots\ldots\,k}=\zeta_{12}\,\zeta_{13}\,\zeta_{23}\,\zeta_{14}\,\zeta_{24}\,\zeta_{34}\ldots\ldots\zeta_{1k}\,\zeta_{2k}\ldots\ldots\zeta_{k-1,\,k},$$

where any symbol $\zeta_{rs}=\varepsilon_{rs}\,.\,\sqrt{b_r-b_s}$, the difference being taken so that b_s, b_r are in ascending order.

Consider now $2n$ indices 1, 2, 3, ..., $2n$; divide them into two sets $r_1,\ldots\ldots,r_n$ and $s_1,s_2,\ldots\ldots,s_n$; let

$$\begin{aligned}D_n&=(b_{r_1}-b_{r_2})\ldots\ldots(b_{r_1}-b_{r_n})(b_{r_2}-b_{r_3})\ldots\ldots(b_{r_2}-b_{r_n})\ldots\ldots(b_{r_{n-1}}-b_{r_n}),\\ E_n&=(b_{s_1}-b_{s_2})\ldots\ldots(b_{s_1}-b_{s_n})(b_{s_2}-b_{s_3})\ldots\ldots(b_{s_2}-b_{s_n})\ldots\ldots(b_{s_{n-1}}-b_{s_n});\end{aligned}$$

then we proceed to show that

$$\vartheta^{(n-1)}(u)\,\vartheta_{12\,\ldots\ldots\,(2n)}(u)=\begin{vmatrix}\phi_{r_1s_1} & \phi_{r_1s_2}\ldots\ldots & \phi_{r_1s_n}\\ \phi_{r_2s_1} & \phi_{r_2s_2}\ldots\ldots & \phi_{r_2s_n}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \phi_{r_ns_1} & \phi_{r_ns_2}\ldots\ldots & \phi_{r_ns_n}\end{vmatrix}\frac{\nabla_{12\,\ldots\ldots\,(2n)}}{D_n\,E_n};$$

and further, taking $(2n+1)$ indices 1, 2, 3, ..., $(2n+1)$, and dividing them into two sets $r_1,\ldots\ldots,r_n$ and $s_1,s_2,\ldots\ldots,s_{n+1}$, and putting

$$D_n=(b_{r_1}-b_{r_2})\ldots\ldots(b_{r_1}-b_{r_n})\ldots\ldots(b_{r_{n-1}}-b_{r_n}),$$

as before, and

$$E_{n+1} = (b_{s_1} - b_{s_2}) \ldots . (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_3}) \ldots .$$
$$\times (b_{s_{n-1}} - b_{s_n})(b_{s_{n-1}} - b_{s_{n+1}})(b_{s^n} - b_{s_{n+1}})$$
$$= E_n (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}}) \ldots . (b_{s_n} - b_{s_{n+1}}),$$

then we proceed to prove that

$$\vartheta^{(n)}(u)\, \vartheta_{123 \ldots . (2n+1)}(u) = \begin{vmatrix} \vartheta_{s_1} & \vartheta_{s_2} & \ldots . \vartheta_{s_{n+1}} \\ \phi_{r_1 s_1} & \phi_{r_1 s_2} & \ldots . \phi_{r_1 s_{n+1}} \\ \ldots & \ldots & \ldots \\ \phi_{r_n s_1} & \phi_{r_n s_2} & \ldots . \phi_{r_n s_{n+1}} \end{vmatrix} \frac{\nabla_{12 \ldots . (2n+1)}}{D_n E_{n+1}};$$

the latter result has been shown to hold for $n = 1$; we deduce that if the latter result holds for any value of n, then the former result holds for the value $n + 1$, and that if the former result holds, the latter result holds; by combining these we have the truth of the theorems in general.

If in the latter result we increase u by $u^{a, b_{r_{n+1}}}$, we obtain, dividing out by a certain exponential factor,

$$\vartheta^{(n)}_{r_{n+1}}(u)\, \vartheta_{123 \ldots . (2n+2)}(u) \frac{D_n E_{n+1}}{\nabla_{12 \ldots . (2n+1)}} =$$
$$\begin{vmatrix} \zeta_{r_{n+1} s_1} \phi_{r_{n+1} s_1} & , \ldots . , \zeta_{r_{n+1} s_{n+1}} \phi_{r_{n+1} s_{n+1}} \\ \zeta_{r_1 r_{n+1}} \zeta_{s_1 r_{n+1}} \frac{1}{\vartheta} \frac{\vartheta_{r_1} \phi_{r_{n+1} s_1} - \vartheta_{r_{n+1}} \phi_{r_1 s_1}}{b_{r_1} - b_{r_{n+1}}}, \ldots . , & \zeta_{r_1 r_{n+1}} \zeta_{r_{n+1} s_{n+1}} \frac{1}{\vartheta} \cdot \frac{\vartheta_{r_1} \phi_{r_{n+1} s_{n+1}} - \vartheta_{r_{n+1}} \phi_{r_1 s_{n+1}}}{b_{r_1} - b_{r_{n+1}}} \\ \ldots & \ldots \\ \zeta_{r_n r_{n+1}} \zeta_{s_1 r_{n+1}} \frac{1}{\vartheta} \frac{\vartheta_{r_n} \phi_{r_{n+1} s_1} - \vartheta_{r_{n+1}} \phi_{r_n s_1}}{b_{r_n} - b_{r_{n+1}}}, \ldots . , & \zeta_{r_n r_{n+1}} \zeta_{r_{n+1} s_{n+1}} \frac{1}{\vartheta} \cdot \frac{\vartheta_{r_n} \phi_{r_{n+1} s_{n+1}} - \vartheta_{r_{n+1}} \phi_{r_n s_{n+1}}}{b_{r_n} - b_{r_{n+1}}} \end{vmatrix}$$

$$= \frac{\zeta_{r_1 r_{n+1}} \ldots . \zeta_{r_n r_{n+1}} \zeta_{r_{n+1} s_1} \ldots . \zeta_{r_{n+1} s_{n+1}}}{(b_{r_1} - b_{r_{n+1}}) \ldots . (b_{r_n} - b_{r_{n+1}})} \frac{\vartheta^{(n)}_{r_{n+1}}}{\vartheta^{(n)}} \begin{vmatrix} \phi_{r_1 s_1} & \ldots . \phi_{r_1 s_{n+1}} \\ \phi_{r_2 s_1} & \ldots . \phi_{r_2 s_{n+1}} \\ \ldots & \ldots \\ \phi_{r_{n+1} s_1} & \ldots . \phi_{r_{n+1} s_{n+1}} \end{vmatrix}$$

and hence

$$\vartheta^{n}(u)\,\vartheta_{123\,\ldots.\,(2n+2)}(u)=\begin{vmatrix}\wp_{r_1s_1} & \ldots.\,\wp_{r_1\,s_{n+1}}\\ \wp_{r_2s_1} & \ldots.\,\wp_{r_2\,s_{n+1}}\\ \ldots\ldots & \ldots\ldots\\ \wp_{r_{n+1}s_1} & \ldots.\,\wp_{r_{n+1}\,s_{n+1}}\end{vmatrix}\,\frac{\nabla_{12\,\ldots.\,(2n+2)}}{D_{n+1}E_{n+1}}\,.$$

Further, assuming

$$\vartheta^{(n-1)}(u)\,\vartheta_{12\,\ldots\,(2n)}(u)\,\frac{D_n\,E_n}{\nabla_{12\,\ldots.\,(2n)}}=\begin{vmatrix}\wp_{r_1s_1}, & \ldots., & \wp_{r_1s_n}\\ \ldots & \ldots & \ldots\\ \wp_{r_ns_1}, & \ldots., & \wp_{r_n\,s_n}\end{vmatrix},$$

we obtain, increasing u by $u^{a,\,b_{s_{n+1}}}$, and dividing out by a certain exponential factor,

$$\vartheta^{(n-1)}_{s_{n+1}}(u)\,\vartheta_{12\,\ldots.\,(2n+1)}(u)\,\frac{D_n\,E_n}{\nabla_{12\,\ldots.\,(2n)}}=$$

$$\begin{vmatrix}\zeta_{r_1\,s_{n+1}}\zeta_{s_1\,s_{n+1}}\dfrac{1}{\vartheta}\,\dfrac{\vartheta_{s_1}\wp_{r_1\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_1s_1}}{b_{s_1}-b_{s_{n+1}}}, & \ldots., & \zeta_{r_1\,s_{n+1}}\zeta_{s_n\,s_{n+1}}\dfrac{1}{\vartheta}\,\dfrac{\vartheta_{s_n}\wp_{r_1\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_1s_n}}{b_{s_n}-b_{s_{n+1}}}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ \zeta_{r_n\,s_{n+1}}\zeta_{s_1\,s_{n+1}}\dfrac{1}{\vartheta}\,\dfrac{\vartheta_{s_1}\wp_{r_n\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_n\,s_1}}{b_{s_1}-b_{s_{n+1}}}, & \ldots, & \zeta_{r_n\,s_{n+1}}\zeta_{s_n\,s_{n+1}}\dfrac{1}{\vartheta}\,\dfrac{\vartheta_{s_n}\wp_{r_ns_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_ns_n}}{b_{s_n}-b_{s_{n+1}}}\end{vmatrix}$$

$$=(-1)^n\,\frac{\zeta_{r_1\,s_{n+1}}\ldots.\zeta_{r_n\,s_{n+1}}\,\zeta_{s_1\,s_{n+1}}\ldots.\zeta_{s_n\,s_{n+1}}}{(b_{s_1}-b_{s_{n+1}})\ldots.(b_{s_n}-b_{s_{n+1}})}\,\frac{1}{\vartheta^n}$$

$$\begin{vmatrix}\vartheta_{s_1} & ,\ldots., & \vartheta_{s_n} & ,\ 1\\ \vartheta_{s_1}\wp_{r_1\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_1\,s_1} & ,\ldots., & \vartheta_{s_n}\wp_{r_1\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_1s_n} & ,\ 0\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\\ \vartheta_{s_1}\wp_{r_n\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_n\,s_1} & ,\ldots., & \vartheta_{s_n}\wp_{r_n\,s_{n+1}}-\vartheta_{s_{n+1}}\wp_{r_n\,s_n} & ,\ 0\end{vmatrix}$$

$$=\frac{\zeta_{r_1\,s_{n+1}}\ldots.\zeta_{r_n\,s_{n+1}}\,\zeta_{s_1\,s_{n+1}}\ldots.\zeta_{s_n\,s_{n+1}}}{(b_{s_1}-b_{s_{n+1}})\ldots.(b_{s_n}-b_{s_{n+1}})}\,\frac{\vartheta^{(n-1)}_{s_{n+1}}}{\vartheta^{(n)}}\begin{vmatrix}\vartheta_{s_1} & ,\ldots., & \vartheta_{s_n} & ,\ \vartheta_{s_{n+1}}\\ \wp_{r_1s_1}, & \ldots., & \wp_{r_1s_n}, & \wp_{r_1\,s_{n+1}}\\ \ldots & \ldots & \ldots & \ldots\\ \wp_{r_ns_1}, & \ldots., & \wp_{r_ns_n}, & \wp_{r_n\,s_{n+1}}\end{vmatrix}$$

so that

$$\vartheta^{n}(u)\,\vartheta_{12\,\ldots\,2n+1}(u)=\begin{vmatrix}\vartheta_{s_1} & ,\ldots., & \vartheta_{s_n} & ,\ \vartheta_{s_{n+1}}\\ \wp_{r_1s_1}, & \ldots., & \wp_{r_1s_n}, & \wp_{r_1\,s_{n+1}}\\ \ldots & \ldots & \ldots & \ldots\\ \wp_{r_ns_1}, & \ldots., & \wp_{r_n\,s_n}, & \wp_{r_n\,s_{n+1}}\end{vmatrix}\,\frac{\nabla_{12\,\ldots.\,(2n+1)}}{D_n\,E_{n+1}}\,.$$

SUMMARY OF SECTION IX.

These results may be summarized as follows: Let $\lambda_1, \lambda_2, \ldots.$ be any quantities whatever; let

$$\vartheta_{123 \ldots. k}(u) = \vartheta(u \,|\, u^{a, b_1} + \ldots. + u^{a, b_k}),$$

where $b_1, \ldots., b_k$ are any finite branch places, and the characteristic is *unreduced;* let $\zeta_{rs} = \varepsilon_{rs}\sqrt{b_r - b_s}$, where the difference is to be taken so that b_s, b_r are in ascending order; let $\nabla_{1 \ldots. k}$ be the product of all the $\frac{1}{2}k(k-1)$ quantities ζ_{rs} in which r, s are a pair of different suffixes chosen from $1, \ldots., k$; then putting

$$\sigma(u) = \frac{\vartheta(u)}{\vartheta(0)},$$

$$\sigma_{12 \ldots. (2n)}(u) = (-1)^n \frac{\lambda_1\lambda_2 \ldots. \lambda_{2n}}{\nabla_{12 \ldots. (2n)}} \frac{\vartheta_{12 \ldots. (2n)}(u)}{\vartheta(0)}, \quad (n = 1, 2 \ldots)$$

$$\sigma_{12 \ldots. (2n+1)}(u) = \frac{\lambda_1\lambda_2 \ldots. \lambda_{2n+1}}{\nabla_{12 \ldots. (2n+1)}} \frac{\vartheta_{12 \ldots. (2n+1)}(u)}{\vartheta(0)}, \quad (n = 0, 1, 2, \ldots)$$

we have

$$\sigma^{(n-1)}(u)\,\sigma_{12 \ldots. (2n)}(u) = \begin{vmatrix} \sigma_{r_1 s_1}(u), & \ldots., & \sigma_{r_1 s_n}(u) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \sigma_{r_n s_1}(u), & \ldots., & \sigma_{r_n s_n}(u) \end{vmatrix} \frac{1}{D_n E_n},$$

and

$$\sigma^{(n)}(u)\,\sigma_{12 \ldots. (2n+1)}(u) = \begin{vmatrix} \sigma_{r_1 s_1}(u), & \ldots., & \sigma_{r_1 s_{n+1}}(u) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \sigma_{r_n s_1}(u), & \ldots., & \sigma_{r_n s_{n+1}}(u) \\ \sigma_{s_1}(u), & \ldots., & \sigma_{s_{n+1}}(u) \end{vmatrix} \frac{1}{D_n E_{n+1}}$$

where, in the former, if $r_1, \ldots., r_n$ and $s_1, \ldots., s_n$ be any decomposition of the indices $1, 2, \ldots., (2n)$ into two sets of n each, then

$$D_n = (b_{r_1} - b_{r_2}) \ldots. (b_{r_1} - b_{r_n})(b_{r_2} - b_{r_3}) \ldots. (b_{r_2} - b_{r_n}) \ldots. (b_{r_{n-1}} - b_{r_n}),$$

$$E_n = (b_{s_1} - b_{s_2}) \ldots. (b_{s_1} - b_{s_n})(b_{s_2} - b_{s_3}) \ldots. (b_{s_2} - b_{s_n}) \ldots. (b_{r_{n-1}} - b_{r_n}),$$

and in the latter, if $r_1, \ldots., r_n$ and $s_1, \ldots., s_{n+1}$ be any decomposition of $1, 2, \ldots., (2n+1)$ into two sets of respectively n and $n+1$ terms, then D_n is as before and

$$E_{n+1} = E_n (b_{s_1} - b_{s_{n+1}})(b_{s_2} - b_{s_{n+1}}) \ldots. (b_{s_n} - b_{s_{n+1}}).$$

In these results, as already stated, $b_1, \ldots., b_{2n+1}$ are any finite branch places, and, correspondingly, $\lambda_1, \ldots., \lambda_{2n+1}$ are arbitrary quantities.

But in the application which we make of the formulæ, $b_1, \ldots, b_{2n+1}$ *will be chosen only from the* p *branch places* $a_1, \ldots, a_p$, *and, correspondingly,* $\lambda_1, \ldots, \lambda_{2n+1}$ *will have the values before assigned to them (Section VI).* It follows then from the form of these equations that, in the expansion of the σ functions in powers of $u_1, \ldots, u_p$, the function $\sigma_{12 \ldots (2n)}(u)$ will begin with terms of degree n, and the function $\sigma_{12 \ldots (2n+1)}(u)$ will begin with terms of degree $(n+1)$. In other words, $\sigma_{12 \ldots k}(u)$ *vanishes to order* $\frac{1}{2}k$ *or* $\frac{1}{2}(k+1)$ *when the arguments* $u_1, \ldots, u_p$ *vanish, and has parity* $(-1)^{\frac{1}{2}k}$, *or* $(-1)^{\frac{1}{2}(k+1)}$ *according as* k *is even or odd.* We shall obtain presently the actual forms of the leading terms in the expansions of the functions, from which it will appear that the functions agree with those considered by Klein. In fact, for $k = 2n$ the function $P(x)/(x-a_1)\ldots(x-a_k)$ is to be identified with that denoted by Klein by $\phi_x^{p+1-2\mu}$, the additional factor being $x-a$ which is here to be replaced by 1; the $\frac{1}{2}k$ is to be identified with the symbol μ used by Klein; in other words, the function $\sigma_{12 \ldots (2n)}(u)$ is that whose algebraical characteristic is given by the decomposition

$$(x-a)\,P(x)/(x-a_1)\ldots(x-a_k); \quad Q(x).(x-a_1)\ldots(x-a_k).$$

Similarly for $k = 2n+1$, we have $\mu = \frac{1}{2}(k+1)$, and $\sigma_{12 \ldots (2n+1)}(u)$ corresponds to the decomposition

$$P(x)/(x-a_1)\ldots(x-a_k); \quad (x-a)\,Q(x)(x-a_1)\ldots(x-a_k).$$

(Compare B. 436).

We now obtain the leading terms in the expansions of each of the functions $\sigma_{12 \ldots (2n)}(u)$, $\sigma_{12 \ldots (2n+1)}(u)$. The deduction is an immediate consequence of the fact (Scott, "Determinants," Cambridge, 1880, p. 121) that the determinant

$$\begin{vmatrix} \dfrac{1}{x_1-a_1}, & \dfrac{1}{x_1-a_2}, & \ldots, & \dfrac{1}{x_1-a_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{x_n-a_1}, & \dfrac{1}{x_n-a_2}, & \ldots, & \dfrac{1}{x_n-a_n} \end{vmatrix}$$

is equal to

$$(-1)^{\frac{1}{2}n(n-1)} \prod_{i<j}\prod (x_i-x_j) \prod_{i<j}\prod (a_i-a_j) \Big/ \prod_{i=1}^{n} (x_i-a_1)\ldots(x_i-a_n).$$

From this we find (cf. Section VIII, Summary) that if $\xi_1, \ldots, \xi_n$ be equivalent symbols, to be afterwards interpreted by the rule

$$\xi_i^{r-1} = u_r, \quad \xi_i^{(0)} = u_1, \qquad (i = 1, \ldots, n;\ r = 2, \ldots, p)$$

and $\phi(x) = P(x)/(x - a_1) \ldots (x - a_{2n})$, then the function $\sigma_{12 \ldots (2n)}(u)$ begins with the terms denoted by

$$\frac{1}{n!} \phi(\xi_1) \ldots \phi(\xi_n) \Delta(\xi_1, \ldots, \xi_n),$$

where $\Delta(\xi_1, \ldots, \xi_n)$ is the product of the squares of the differences of the quantities $\xi_1, \ldots, \xi_n$. While similarly the function $\sigma_{12 \ldots (2n+1)}(u)$ begins with the terms denoted by

$$\frac{1}{(n+1)!} \phi(\xi_1) \ldots \phi(\xi_{n+1}) \Delta(\xi_1, \ldots, \xi_{n+1}),$$

where $\phi(x) = P(x)/(x - a_1) \ldots (x - a_{2n+1})$. Cf. Burkhardt, Math. Annal., XXXII (1888), p. 442.

SECTION X.

Expression of the square of a theta function of three or more suffixes in terms of the squares of the functions of one and two suffixes.

The results of the present section are conveniently representable in terms of the algebraical expressions known as Pfaffians, in regard to which, therefore, some preliminary remarks should be made (cf. e. g. Scott, "Determinants," Cambridge, 1880, p. 74).

Let (12) denote any algebraical quantity, and $(21) = -(12)$, and let $(12 \ldots k)$ denote the integral rational expression, of order k in the quantities (rs), which is defined by the equation

$$\begin{aligned}(12 \ldots k) &= (12)(34 \ldots k) - (13)(245) \ldots k) + (14)(235 \ldots k) \\ &\qquad - \ldots + (1k)(234 \ldots \overline{k-1}) \\ &= (12)(34) \ldots (\overline{k-1k}) \pm \ldots,\end{aligned}$$

where the other terms of the latter form are obtainable from the one written down by interchange of the indices 2, 3, 4, ..., with proper changes of sign. We suppose k to be even, $= 2n$; when k is odd the function vanishes identically. The square of the function is equal to the determinant whose $(i, j)^{\text{th}}$ element is (ij) when $i < j$, and $-(ji)$ when $i > j$.

In regard to this expression we utilize the two results:

(I). If in the expanded result every element such as (12) be replaced by $(xy12)$, that is, by

$$(xy)(12) - (x1)(y2) + (x2)(y1),$$

x and y being indices other than 1, 2, ..., $2n$, the result will be

$$(xy)^{n-1}(xy1234 \ldots . \overline{2n-1} \,.\, 2n),$$

namely, we have

$$(xy12)(xy34) \ldots (xy\overline{2n-1} \,.\, 2n) \pm \ldots = (xy)^{n-1}(xy1234 \ldots \overline{2n-1} \,.\, 2n). \quad \text{(I)}$$

For instance, n being 2, it is easily verified by direct calculation that

$$(xy12)(xy34) - (xy13)(xy24) + (xy14)(xy23) = (xy)(xy1234).$$

(II). If in the expanded expression of the Pfaffian $(y123 \ldots . \overline{2n+1})$, namely,

$$(y1)(23)(45) \ldots . (2n \,.\, \overline{2n+1}) \pm \ldots .,$$

we replace every element whose symbol does not contain the index y, such as (23), by a Pfaffian of four indices $(xy23)$, and replace every element involving y, such as $(y1)$, by the corresponding element in which x replaces y, namely $(x1)$, the result will be

$$(xy)^n(x123 \ldots . \overline{2n+1}),$$

namely, we have

$$(x1)(xy23)(xy45) \ldots . (xy2n \,.\, \overline{2n+1}) \pm \ldots . = (xy)^n(x123 \ldots . \overline{2n+1}). \quad \text{(II)}$$

For instance, when $n = 1$, we have, as is easily verified,

$$(x1)(xy23) - (x2)(xy13) + (x3)(xy12) = (xy)(x123).$$

To prove (I), notice that the expansion

$$[(xy)(12) - (x1)(y2) + (x2)(y1)] \ldots . [(xy)(x\beta) - (x\alpha)(y\beta) + (x\beta)(y\alpha)] \ldots .$$
$$[(xy)(2n-1 \,.\, 2n) - (x2n-1)(y2n) + (x2n)(y2n-1)] \pm \ldots .$$

is

$$\begin{aligned} & (xy)^n(123 \ldots . 2n) \\ & + (xy)^{n-1}(12)(34) \ldots . (2n-1\ 2n)(x\beta)(y\alpha) \pm \ldots . \\ & + (xy)^{n-2}(12)(34) \ldots . (2n-1\ 2n)(x\beta)(y\alpha)(x\beta')(y\alpha') \pm \ldots . \\ & + \ldots . \\ & + (x\beta_1)(x\beta_2) \ldots . (x\beta_n)(y\alpha_1) \ldots . (y\alpha_n) \pm \ldots ., \end{aligned}$$

wherein, in the second line, the product $(12)(34)\ldots\ldots(2n-1\ 2n)$ does not contain the factor $(\alpha\beta)$, and in the third line the product $(12)(34)\ldots\ldots(2n-1\ 2n)$ does not contain the terms $(\alpha\beta)$, $(\alpha'\beta')$, and so on.

Now the aggregate of the terms in the last line, which are those in which no factor (xy) enters, is zero; for to every term $(x\beta_1)\ldots\ldots(x\beta_n)(y\alpha_1)\ldots\ldots(y\alpha_n)$ we may make correspond another term, also occurring in this line, which differs from this one only in the interchange of two of the indices $\beta_1, \ldots\ldots, \beta_n$; on account of this single interchange this corresponding term will have a different sign from that of the term written down, and the two terms will cancel one another.

A similar argument applies to every line of the expansion except the first and second; for instance, in the third line corresponding to the term

$$(xy)^{n-2}(12)(34)\ldots\ldots(2n-1\ 2n)(x\beta)(y\alpha)(x\beta')(y\alpha')$$

there is a term

$$(xy)^{n-2}(12)(34)\ldots\ldots(2n-1\ 2n)(x\beta')(y\alpha)(x\beta)(y\alpha'),$$

which, on account of the interchange of β and β', has a different sign from that of the term written down.

The first and second lines together clearly give

$$\begin{aligned}&(xy)^{n-1}\left[(xy)(123\ldots\ldots 2n)-(x1)(y23\ldots\ldots 2n)+(x2)(y13\ldots\ldots 2n)-\ldots\ldots\right]\\ &=(xy)^{n-1}(xy12\ldots\ldots 2n),\end{aligned}$$

which proves the theorem.

The result (II) can be deduced from (I). In virtue of that theorem, the expression on the left side of equation (II) is in fact

$$\begin{aligned}&(x1)(xy)^{n-1}(xy23\ldots\ldots 2n+1)-(x2)(xy)^{n-1}(xy13\ldots\ldots 2n+1)+\ldots\ldots\\ &\qquad\qquad+(x2n+1)(xy)^{n-1}(xy12\ldots\ldots 2n)\\ &=(xy)^{n-1}\left[(x1)(xy23\ldots\ldots 2n+1)-(x2)(xy13\ldots\ldots 2n+1)+\ldots\ldots\right.\\ &\qquad\qquad\left.+(x2n+1)(xy12\ldots\ldots 2n)\right]\\ &=(xy)^{n-1}\left[(xxy12\ldots\ldots 2n+1)+(xy)(x123\ldots\ldots 2n+1)\right]\\ &=(xy)^{n}(x123\ldots\ldots 2n+1),\ \text{since}\ (xxy12\ldots\ldots 2n+1)=0;\end{aligned}$$

and this proves the result in question.

In the applications now to be made of these theorems (I) and (II), we put $\vartheta_{rs}(u)$ to denote $\vartheta\left(u\,|\,u^{a,\,b_r}+u^{a,\,b_s}\right)$, where b_r, b_s are any two finite branch places, and, more generally, $\vartheta_{12\ldots k}(u)$ to denote $\vartheta(u\,|\,u^{a_1, b_1}+\ldots+u^{a_k, b_k})$, and we put, for the elements (rs) of the Pfaffians which occur,

$$\begin{aligned} \text{when } r<s,\ (rs) &= \vartheta^2_{rs}(u), \text{ with } (0s) = \vartheta^2_s(u), \\ r>s,\ (rs) &= -\vartheta^2_{rs}(u), \text{ with } (r0) = -\vartheta^2_r(u); \end{aligned}$$

for the sake of brevity we replace $\vartheta^2_{rs}(u)$ simply by ϑ^2_{rs}, and $\vartheta^2_r(u)$ by ϑ^2_r.

The results to be obtained are those represented by the equations

$$\begin{aligned} \vartheta^{2(n-1)}(u)\,\vartheta^2_{12\ldots 2n}(u) &= (12\ldots 2n), \\ \vartheta^{2n}(u)\,\vartheta^2_{12\ldots 2n+1}(u) &= (012\ldots 2n+1), \end{aligned} \qquad (n=1,\,2,\,3,\ldots) \qquad \text{(III)}$$

which hold for the dissection (I) of the Riemann surface by means of which the theta functions are defined.

The method of proof is by induction. We have already proved (Section VII) that

$$\vartheta^2(u)\,\vartheta^2_{123}(u) = \vartheta^2_1(u)\,\vartheta^2_{23}(u) - \vartheta^2_2(u)\,\vartheta^2_{13}(u) + \vartheta^2_3(u)\,\vartheta^2_{12}(u),$$

or, as we write it,

$$\vartheta^2\vartheta^2_{123} = (0123);$$

we assume that

$$\vartheta^{2n}\,\vartheta^2_{12\ldots(2n+1)} = (012\ldots 2n\,.\,2n+1),$$

and we deduce that

$$\vartheta^{2n}\,\vartheta^2_{12\ldots(2n+2)} = (123\ldots 2n+1\,,\,2n+2),$$

and

$$\vartheta^{2n+2}\,\vartheta^2_{12\ldots(2n+3)} = (012\ldots 2n+2\,.\,2n+3);$$

these suffice to prove the results (III) in all cases.

It is worth while to illustrate the nature of the proof in detail by deducing from $\vartheta^2\vartheta^2_{123} = (0123)$ the next following case, namely, the equation

$$\vartheta^2\vartheta^2_{1234} = \vartheta^2_{12}\vartheta^2_{34} - \vartheta^2_{13}\vartheta^2_{24} + \vartheta^2_{14}\vartheta^2_{23}.$$

We have, as cases of the equation $\vartheta^2\vartheta^2_{123} = (0123)$, the following, where 1, 2, 3, 4 are associated with any four finite branch places in ascending order, namely,

$$\begin{aligned} \vartheta^2\vartheta^2_{234} &= \vartheta^2_2\vartheta^2_{34} - \vartheta^2_3\vartheta^2_{24} + \vartheta^2_4\vartheta^2_{23}, \\ \vartheta^2\vartheta^2_{134} &= \vartheta^2_1\vartheta^2_{34} - \vartheta^2_3\vartheta^2_{14} + \vartheta^2_4\vartheta^2_{13}, \\ \vartheta^2\vartheta^2_{124} &= \vartheta^2_1\vartheta^2_{24} - \vartheta^2_2\vartheta^2_{14} + \vartheta^2_4\vartheta^2_{12}; \end{aligned}$$

from these we deduce

$$\vartheta^2(\vartheta_{14}^2\vartheta_{234}^2-\vartheta_{24}^2\vartheta_{134}^2+\vartheta_{34}^2\vartheta_{124}^2)=\vartheta_4^2(\vartheta_{12}^2\vartheta_{34}^2-\vartheta_{13}^2\vartheta_{24}^2+\vartheta_{14}^2\vartheta_{23}^2)\,;$$

but from $\vartheta^2\vartheta_{123}^2=(0123)$, by increasing the argument by the half-period $u^{a,\,b_4}$, we deduce

$$\vartheta_4^2\vartheta_{1234}^2=\vartheta_{14}^2\vartheta_{234}^2-\vartheta_{24}^2\vartheta_{134}^2+\vartheta_{34}^2\vartheta_{124}^2\,,$$

hence the result in question follows. In other words, the functions $\vartheta^2(u)\,\vartheta_{123}^2(u)$, $\vartheta^2(u)\,\vartheta_{1234}^2(u)$ are square roots respectively of the determinants

$$\begin{vmatrix} 0\,, & \vartheta_1^2\,, & \vartheta_2^2\,, & \vartheta_3^2 \\ -\vartheta_1^2, & 0\,, & \vartheta_{12}^2, & \vartheta_{13}^2 \\ -\vartheta_2^2, & -\vartheta_{12}^2, & 0\,, & \vartheta_{23}^2 \\ -\vartheta_3^2, & -\vartheta_{13}^2, & -\vartheta_{23}^2, & 0 \end{vmatrix}, \quad \begin{vmatrix} 0\,, & \vartheta_{12}^2, & \vartheta_{13}^2, & \vartheta_{14}^2 \\ -\vartheta_{12}^2, & 0\,, & \vartheta_{23}^2, & \vartheta_{24}^2 \\ -\vartheta_{13}^2, & -\vartheta_{23}^2, & 0\,, & \vartheta_{34}^2 \\ -\vartheta_{14}^2, & -\vartheta_{24}^2, & -\vartheta_{34}^2, & 0 \end{vmatrix}.$$

Similarly, for the next case, we deduce from $\vartheta^2\vartheta_{1234}^2=(1234)$ that

$$\vartheta_5^2\vartheta_{12345}^2=\vartheta_{125}^2\vartheta_{345}^2-\vartheta_{135}^2\vartheta_{245}^2+\vartheta_{145}^2\vartheta_{235}^2\,,$$

and hence, by substitution of the Pfaffian expressions for $\vartheta^2\vartheta_{125}^2$, etc., that $\vartheta^4\vartheta_{12345}^2=(012345)$, which is a square root of the determinant

$$\begin{vmatrix} 0\,, & \vartheta_1^2\,, & \vartheta_2^2\,, & \vartheta_3^2\,, & \vartheta_4^2\,, & \vartheta_5^2 \\ -\vartheta_1^2, & 0\,, & \vartheta_{12}^2, & \vartheta_{13}^2, & \vartheta_{14}^2, & \vartheta_{15}^2 \\ -\vartheta_2^2, & -\vartheta_{12}^2, & 0\,, & \vartheta_{23}^2, & \vartheta_{24}^2, & \vartheta_{25}^2 \\ -\vartheta_3^2, & -\vartheta_{13}^2, & -\vartheta_{23}^2, & 0\,, & \vartheta_{34}^2, & \vartheta_{35}^2 \\ -\vartheta_4^2, & -\vartheta_{14}^2, & -\vartheta_{24}^2, & -\vartheta_{34}^2, & 0\,, & \vartheta_{45}^2 \\ -\vartheta_5^2, & -\vartheta_{15}^2, & -\vartheta_{25}^2, & -\vartheta_{35}^2, & -\vartheta_{45}^2, & 0 \end{vmatrix}.$$

We pass now to the proof of the general formulæ (III); if in the equation $\vartheta^{2n}\vartheta_{12\,\ldots\,(2n+1)}^2=(012\,\ldots\,2n\;2n+1)$, namely,

$$\vartheta^{2n}\vartheta_{12\,\ldots\,(2n+1)}^2=\vartheta_1^2\vartheta_{23}^2\ldots\vartheta_{2n\,.\,2n+1}^2\pm\ldots\,,$$

we increase the argument by the half-period $u^{a,\,b_{2n+2}}$, we obtain

$$\vartheta^{2n}\vartheta_{2n+2}^{2n}\vartheta_{12\,\ldots\,(2n+2)}^2=\vartheta_{1\,2n+2}^2\,.\,\vartheta^2\vartheta_{23\,2n+2}^2\ldots\vartheta^2\vartheta_{2n\,.\,2n+1\,.\,2n+2}^2\pm\ldots\,;$$

here the right side is

$$\begin{aligned}&(1\,.\,2n+2)(023\;2n+2)\ldots(0\,.\,2n\,.\,2n+1\,.\,2n+2)\pm\ldots,\\ &=[(x1)(xy23)(xy45)\ldots(xy\;2n\;2n+1)\pm\ldots](-1)^{n+1},\end{aligned}$$

where x is put for $2n+2$ and y for 0; by the theorem (II) of this section this is equal to

$$\begin{aligned} &(-1)^{n+1}(xy)^n(x123\ldots.2n+1),\\ =\;&(0\;2n+2)^n(123\ldots.2n+2),\\ =\;&\vartheta^{2n}_{2n+2}(123\ldots.2n+2); \end{aligned}$$

thus

$$\vartheta^{2n}\vartheta^2_{12\ldots.(2n+2)}=(123\ldots.2n+2). \qquad (\alpha)$$

Replacing, in the last obtained equation, n by $m-1$ for convenience of writing, we have

$$\vartheta^{2(m-1)}\vartheta^2_{12\ldots.2m}=\vartheta^2_{12}\vartheta^2_{34}\ldots.\vartheta^2_{2m-1.2m}\pm\ldots.,$$

and hence, adding a half-period $u^{a,\,b_{2m+1}}$ to the argument,

$$\begin{aligned} \vartheta^{2m}\vartheta^{2(m-1)}_{2m+1}\vartheta^2_{12\ldots.(2m+1)}&=\vartheta^2\vartheta^2_{12.2m+1}\vartheta_2\vartheta^2_{34\,2m+1}\ldots.\vartheta^2\vartheta^2_{2m-1.2m.2m+1}\pm\ldots,\\ &=(012\;2m+1)(034\;2m+1)\ldots(0\;2m-1\;2m\;2m+1)\pm\ldots,\\ &=(xy12)(xy34)\ldots(xy\;2m-1\;2m)\pm\ldots, \end{aligned}$$

where x, y replace 0 and $2m+1$ respectively; by the theorem (I) of this section the right side is equal to

$$\begin{aligned} &(xy)^{m-1}(xy1234\ldots.2m-1.2m)\\ =\;&(0\;2m+1)^{m-1}(01234\ldots.2m.2m+1)\\ =\;&\vartheta^{2(m-1)}_{2m+1}.(012\ldots.2m.2m+1); \end{aligned}$$

hence, dividing by $\vartheta^{2(m-1)}_{2m+1}$ and replacing m by $n+1$, we have

$$\vartheta^{2(n+1)}\vartheta^2_{12\ldots.(2n+3)}=(012\ldots.2n+2.2n+3). \qquad (\beta)$$

Now as the first of the equations (III) is identical when $n=1$, and the second of these equations holds for $n=1$, it follows from the equation (α) that the first of the equations (III) holds when $n=2$, and thence, from the equation (β), that the second of the equations (III) holds for $n=2$, and so on.

Two remarks are to be made:

In the first place the two results in equation (III) are not to be regarded as independent; since if in $\vartheta^2_{12\ldots.(2n+1)}(u)$ we suppose the branch place b_{2n+1} to be the branch place a, at infinity, the function becomes the function $\vartheta^2_{12\ldots2n}(u)$, while similarly the results for smaller values of n are to be regarded as particular cases of the results for larger values. (Cf. B. 473, 525.)

In the next place, in the applications which are here to be made of the equation (III), it will be supposed that the branch places $b_1, b_2, \ldots.$ are chosen from among the p places $a_1, \ldots., a_p$.

SECTION XI.

Proof of an addition equation for hyperelliptic theta functions.

The equation in question is given by Königsberger, Crelle, LXIV (see also B. 457). Let B_r denote the characteristic associated with the half-period $u^{a,\, b_r}$, b_r being one of the $2p+1$ finite branch places; let A be an arbitrary half integer characteristic. Denote the 2^p characteristics A, $AB_1 \ldots. B_k$, formed by adding the characteristic A to any combination of 0, 1, 2, ..., p of the characteristics $B_1, \ldots., B_p$ by $Q_1, Q_2, \ldots., Q_s$, where $s = 2^p$. Then if C be the characteristic associated with the half-period $u^{b_1,\, a_1} + \ldots. + u^{b_p,\, a_p}$, every one of the quantities $\vartheta(0 \mid CQ_i Q_j)$, where i is not equal to j, vanishes in the hyperelliptic case, though the characteristic $CQ_i Q_j$ is not necessarily odd.

For

$$
\begin{aligned}
(CQ_i Q_j &\equiv u^{b_1,\, a_1} + \ldots. + u^{b_p,\, a_p} + u^{b_1,\, a} + \ldots. + u^{b_k,\, a}, \qquad (0 < k \not> p) \\
&\equiv u^{a_1,\, a} + \ldots. + u^{a_k,\, a} + u^{b_{k+1},\, a_{k+1}} + \ldots. + u^{b_p,\, a_p}, \\
&\equiv u^{a_p,\, a} - u^{x_1,\, a_1} - \ldots. - u^{x_{p-1},\, a_{p-1}}, \\
&\equiv u^{x_1,\, a_1} + \ldots. + u^{x_{p-1},\, a_{p-1}} + u^{a,\, a_p},
\end{aligned}
$$

provided

$$(x_1, \ldots., x_{p-1}, a) \equiv (a, a, \ldots., a, b_{k+1}, \ldots., b_p),$$

wherein, in the bracket on the right hand, the number of letters a is k, and is not zero. In other words, by taking $x_1, \ldots., x_{p-1}$ respectively equal to $a_1, \ldots., a, b_{k+1}, \ldots., b_p$—where the number of letters a is $k-1$—we can put the half-period with which the characteristic $CQ_i Q_j$ is associated, into a form differing only by periods from

$$u^{a_p,\, a} - u^{x_1,\, a_1} - \ldots. - u^{x_{p-1},\, a_{p-1}};$$

this proves (B. 258) that the function $\vartheta(0 \mid CQ_i Q_j)$ vanishes.

Now, since $2^p + 1$ theta functions of the second order and of characteristic zero are connected by a linear equation, there exists an equation

$$a\,\vartheta(u+v)\,\vartheta(u-v) = \sum_{\lambda=1}^{s} a_\lambda \vartheta^2(u \mid Q_\lambda),$$

wherein $a, a_1, \ldots, a_s$ are constants. In this equation increase u by the period $\Omega_C + \Omega_{Q_r}$; then it becomes

$$a\vartheta(u+v \mid C+Q_r)\,\vartheta(u-v \mid C+Q_r) = \sum_{\lambda=1}^{s} a_\lambda e^{-4\pi i (C'+Q'_r)Q_\lambda}\,\vartheta^2(u \mid C+Q_r+Q_\lambda);$$

therefore, putting $u=0$, we deduce

$$\frac{a_r}{a} = \frac{\vartheta^2(v \mid C+Q_r)\,e^{4\pi i(C'+Q'_r)(C+Q_r)}}{\vartheta^2(0 \mid C+2Q_r)\,e^{-4\pi i(C'+Q'_r)Q_r}},$$

and therefore, *assuming that* $\vartheta(0 \mid u^{b_1, a_1} + \ldots + u^{b_p, a_p})$ *is not zero,*

$$\frac{a_r}{a} = e^{\pi i |C| + \pi i |C, Q_r|}\,\frac{\vartheta^2(v \mid C+Q_r)}{\vartheta^2(0 \mid C)},$$

so that

$$e^{\pi i |C|}\,\vartheta^2(0 \mid C)\,\vartheta(u+v)\,\vartheta(u-v) = \sum_{\lambda=1}^{s} e^{\pi i |C'Q_r|}\,\vartheta^2(u \mid Q_r)\,\vartheta^2(v \mid CQ_r),$$

which is the addition equation in question. In regard to the assumption made, the half-period

$$u^{b_1, a_1} + \ldots + u^{b_p, a_p}$$

can be congruent to an expression of the form

$$u^{x_1, a_1} + \ldots + u^{x_{p-1}, a_{p-1}} + u^{a, a_p}$$

only if a be among the set $b_1, \ldots, b_p$; as we have excluded this possibility, the function $\vartheta(0 \mid C)$ does not vanish (B. 260).

In what follows we consider only the case when the branch places $b_1, \ldots, b_p$ are the p places $a_1, \ldots, a_p$; and we take the characteristic A to be zero; then the two results we have obtained are:

(i). Let A_i denote the half integer characteristic associated with the half-period u^{a, a_i}, and consider the 2^p characteristics

$$0,\ A_1, \ldots, A_p,\ A_1A_2, \ldots, A_{p-1}A_p, \ldots, A_1A_2 \ldots A_p;$$

if Q denote any one of these characteristics, other than the first, the function $\vartheta(u \mid Q)$ vanishes for zero values of the argument, though the characteristic Q is not necessarily odd. This result holds for any *hyperelliptic* case.

(ii). We have, in the hyperelliptic case,

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)=\Sigma\,\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q),$$

where the summation extends to the system of 2^p characteristics enumerated in (i).

SECTION XII.

Proof that $\vartheta(u+v)\,\vartheta(u-v)/\vartheta^2(u)\,\vartheta^2(v)$ *is expressible as an integral polynomial in each of the sets of* $\frac{1}{2}p(p+1)$ *functions*

$$\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u),\qquad \frac{\partial^2}{\partial v_i\,\partial v_j}\log\vartheta(v),$$

and deduction of a set of 2^p *formulæ of this kind.*

We put

$$\wp_{ij}(u)=-\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u),$$

and, in the equation, proved in Section XI,

$$\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)=\vartheta^2(u)\,\vartheta^2(v)+\Sigma\,\vartheta_i^2(u)\,\vartheta_i^2(v)+\Sigma\,\vartheta_{ij}^2(u)\,\vartheta_{ij}^2(v)$$
$$+\,\Sigma\,\vartheta_{ijk}^2(u)\,\vartheta_{ijk}^2(v)+\ldots.,$$

wherein $\vartheta_i(u)$ denotes $\vartheta(u\,|\,u^{a,\,a_i})$, etc., we suppose $u_1,\ldots.,u_p$ to be small, and expand both sides in powers of $u_1,\ldots.,u_p$, and then equate coefficients of the same powers of $u_1,\ldots.,u_p$, on the two sides, up to the second powers of these quantities. On the right side the functions of three or more suffixes do not give any powers of $u_1,\ldots.,u_p$ below the fourth (Section IX), and therefore the coefficients in the expansion of such functions do not enter into the resulting equations. Supposing that

$$\begin{aligned}\frac{\vartheta(u)}{\vartheta(0)}&=1+\tfrac{1}{2}(c_{11}u_1^2+\ldots.+2c_{12}u_1u_2+\ldots.)+\ldots.,\\ \frac{\vartheta_i(u)}{\vartheta(0)}&=A_i u_1+\ldots.+P_i u_p+\ldots.,\\ \frac{\vartheta_{ij}(u)}{\vartheta(0)}&=A_{ij}u_1+\ldots.+P_{ij}u_p+\ldots.,\end{aligned}\qquad (i,j=1,2,\ldots.,p)$$

the developed equation, as far as it involves second powers of $u_1, \ldots, u_p$, is

$$1 - \Sigma\Sigma\, u_i u_j \wp_{ij}(v) = 1 + \Sigma\Sigma c_{ij} u_i u_j \\ + \Sigma\,(A_i u_1 + \ldots + P_i u_p)^2 \frac{\vartheta_i^2(v)}{\vartheta^2(v)} + \Sigma\,(A_{ij} u_1 + \ldots + P_{ij} u_p)^2 \frac{\vartheta_{ij}^2(v)}{\vartheta^2(v)},$$

and by equating the coefficients we have the following $\frac{1}{2}p(p+1)$ equations for the determination of the $\frac{1}{2}p(p+1)$ quotients

$$q_i^2(v) = \vartheta_i^2(v)/\vartheta^2(v), \quad q_{ij}^2(v) = \vartheta_{ij}^2(v)/\vartheta^2(v), \qquad (i, j = 1, 2, \ldots, p)$$

as integral polynomials in the quantities $\wp_{ij}(v)$,

$$\begin{array}{lll} -c_{11} & -\wp_{11}(v) & = \sum\limits_i A_i^2 q_i^2(v) + \sum\limits_{i,j}\sum A_{ij}^2 q_{ij}^2(v), \\ -c_{22} & -\wp_{22}(v) & = \sum\limits_i B_i^2 q_i^2(v) + \sum\limits_{i,j}\sum B_{ij}^2 q_{ij}^2(v), \\ -c_{12} & -\wp_{12}(v) & = \sum\limits_i A_i B_i q_i^2(v) + \sum\limits_{i,j}\sum A_{ii} B_{ii} q_{ii}^2(v), \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ -c_{pp} & -\wp_{pp}(v) & = \sum\limits_i P_i^2 q_i^2(v) + \sum\limits_{i,j}\sum P_{ij}^2 q_{ij}^2(v), \\ -c_{p,\,p-1} & -\wp_{p,\,p-1}(v) & = \sum\limits_i P_i N_i q_i^2(v) + \sum\limits_{i,j}\sum P_{ij} N_{ij} q_{ij}^2(v), \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ -c_{p,1} & -\wp_{p,1}(v) & = \sum\limits_i P_i A_i q_i^2(v) + \sum\limits_{i,j}\sum P_{ij} A_{ij} q_{ij}^2(v); \end{array}$$

the solution of these equations is given in the next section.

Substituting in the addition equation of Section XI the values for $q_i^2(u)$, $q_i^2(v)$, $q_{ij}^2(u)$, $q_{ij}^2(v)$, which result from these equations, it takes the form

$$\frac{\vartheta^2(0)\,\vartheta(u+v)\,\vartheta(u-v)}{\vartheta^2(u)\,\vartheta^2(v)} = 1 + (\wp_{ij}(u),\ \wp_{ij}(v)) \\ + \Sigma\, \frac{\vartheta_{ijk}^2(u)}{\vartheta^2(u)} \cdot \frac{\vartheta_{ijk}^2(v)}{\vartheta^2(v)} + \ldots,$$

wherein $(\wp_{ij}(u),\ \wp_{ij}(v))$ denotes an integral polynomial, linear in each of the two sets of $\frac{1}{2}p(p+1)$ functions $\wp_{ij}(u)$, $\wp_{ij}(v)$; but, now, it follows from the formulæ (Section X)

$$\begin{array}{ll} \vartheta^{2(n-1)}(u)\,\vartheta_{12\ldots 2n}^2(u) & = \vartheta_{12}^2\,\vartheta_{34}^2 \ldots \vartheta_{2n-1\,2n}^2 \pm \ldots, \\ \vartheta^{2n}(u)\quad \vartheta_{12\ldots 2n+1}^2(u) & = \vartheta_1^2\ \vartheta_{23}^2 \ldots \vartheta_{2n.\,2n+1}^2 \pm \ldots, \end{array}$$

that the quotients $q^2_{ijk\,\ldots}(u)$, of three or more suffixes, are expressible as integral polynomials in the quotients $q^2_i(u)$, $q^2_{ij}(u)$ of one and two suffixes, and therefore as integral polynomials in the $\frac{1}{2}p(p+1)$ functions $\wp_{ij}(u)$.

Hence *the function* $\vartheta(u+v)\,\vartheta(u-v)/\vartheta^2(u)\,\vartheta^2(v)$ *is expressible as an integral polynomial in the* $p(p+1)$ *quantities* $\wp_{ij}(u)$, $\wp_{ij}(v)$.

We pass to a further consequence. Suppose that in the addition formula of Section XI we increase the argument u by Ω_Q, where Q is the half integer characteristic associated with a half-period of the form

$$u^{a,\,a_1}+\ldots\ldots+u^{a,\,a_k};$$

then the formula becomes

$$\vartheta^2(0)\,\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)=\vartheta^2(u\,|\,Q)\,\vartheta^2(v)+\Sigma\vartheta^2(u\,|\,QQ_i)\,\vartheta^2(v\,|\,Q_i),$$

where Q_i becomes in turn every one of the 2^p-1 characteristics formed by combinations of $1,\ 2,\ldots\ldots,p$ of the fundamental characteristics $A_1,\ldots\ldots,A_p$; hence

$$\frac{\vartheta^2(0)\,\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)}=\frac{\vartheta^2(v)}{\vartheta^2(v\,|\,Q)}+\Sigma\,\frac{\vartheta^2(u\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)}\cdot\frac{\vartheta^2(v\,|\,Q_i)}{\vartheta^2(v\,|\,Q)};$$

here, on the right side, every theta quotient which occurs is of one of the forms

$$\frac{\vartheta^2(u\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)},\quad\frac{\vartheta^2(v\,|\,QQ_i)}{\vartheta^2(u\,|\,Q)},$$

where Q_i is one of the group of 2^p characteristics previously considered; now we have shown that every quotient $\vartheta^2(u\,|\,Q_i)/\vartheta^2(u)$ is expressible as an integral polynomial in the functions $\partial^2\log\vartheta(u)/\partial u_i\,\partial u_j$; it follows that every quotient $\vartheta^2(u\,|\,QQ_i)/\vartheta^2(u\,|\,Q)$ is expressible as an integral polynomial in the functions

$$\wp_{ij}(u\,|\,Q)=-\frac{\partial^2}{\partial u_i\,\partial u_j}\log\vartheta(u\,|\,Q);$$

we have therefore the result:

Let Q *be any one of the group of* 2^p *characteristics*

$$0,\ A_1,\ldots\ldots,A_p,\ A_1A_2,\ldots\ldots,A_1A_2\ldots\ldots A_k,\ldots\ldots,$$

before considered; let $\wp_{ij}(u\,|\,Q)=-\partial^2\log\vartheta(u\,|\,Q)/\partial u_i\,\partial u_j$; *then the function*

$$\frac{\vartheta(u+v\,|\,Q)\,\vartheta(u-v\,|\,Q)}{\vartheta^2(u\,|\,Q)\,\vartheta^2(v\,|\,Q)}$$

is expressible as an integral polynomial in the $p(p+1)$ *functions* $\wp_{ij}(u \mid Q)$, $\wp_{ij}(v \mid Q)$.

SECTION XIII.

Explicit determination of the $\frac{1}{2}p(p+1)$ *theta quotients* $q_i^2(u)$, $q_{ij}^2(u)$, *in terms of functions* $\wp_{ij}(u)$.

To solve the equations of Section XII we notice that, as follows from Section VIII, the dissection of the Riemann surface being (I), we have

$$\begin{aligned} &A_r + a_i B_r + \ldots\ldots + a_i^{p-1} P_r \\ &\qquad = \frac{1}{\lambda_r}\left[\chi_{p-1}(a_r) + a_i \chi_{p-2}(a_r) + \ldots\ldots + a_i^{p-2}\chi_1(a_r) + a_i^{p-1}\right], \\ &\qquad = 0, \text{ when } i \neq r, \\ &\qquad = \frac{1}{\lambda_r} P'(a_r) = i\lambda_r \sqrt{f'(a_r)/4}, \text{ when } i = r, \end{aligned}$$

and

$$\begin{aligned} &A_{rs} + a_i B_{rs} + \ldots\ldots + a_i^{p-1} P_{rs} \\ &= -\frac{\zeta_{rs}}{\lambda_r \lambda_s}\left[\psi_{p-1}(a_r, a_s) + a_i \psi_{p-2}(a_r, a_s) + \ldots + a_i^{p-2}\psi_1(a_r, a_s) + a_i^{p-1}(0)\right], \\ &= 0, \text{ when } i \neq r,\ i \neq s, \\ &= -\frac{\zeta_{rs}}{\lambda_r \lambda_s}\frac{P'(a_r)}{a_r - a_s}, \text{ when } i = r,\ r \neq s; \end{aligned}$$

therefore, taking the last p of the equations before given (p. 369), and remembering that $P_{rs} = 0$ for every value of r and s, we have

$$\begin{aligned} &-\left\{[c_{pp} + \wp_{pp}(u)]\, a_r^{p-1} + [c_{p\,p-1} + \wp_{p,\,p-1}(u)]\, a_r^{p-2} + \ldots\ldots + [c_{p_1} + \wp_{p,\,1}(u)]\right\} \\ &= P_r\, i\lambda_r \sqrt{f'(a_r)/4}\, q_r^2(u) \\ &= i\sqrt{f'(a_r)/4}\,.\, q_r^2(u); \end{aligned}$$

also, from the same equations,

$$
\begin{aligned}
&-\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{ij}+\wp_{ij}(u)]\,a_r^{i-1}a_s^{j-1}\\
&=\sum_{i=1}^{p}[A_i+a_r B_i+\ldots\ldots+a_r^{p-1}P_i][A_i+a_s B_i+\ldots\ldots+a_s^{p-1}P_i]\,q_i^2(u)\\
&\quad+\sum_{i=1}^{p}\sum_{j=1}^{p}[A_{ij}+a_r B_{ij}+\ldots\ldots+a_r^{p-1}P_{ij}][A_{ij}+a_s B_{ij}+\ldots\ldots+a_s^{p-1}P_{ij}]\,q_{ij}^2(u),\\
&=-\frac{\zeta_{rs}^2}{(a_r-a_s)^2}\,\frac{P'(a_r)\,P'(a_s)}{\lambda_r^2\lambda_s^2}\,q_{r,s}^2(u),\\
&=\frac{\sqrt{f'(a_r)/4}\,\sqrt{f'(a_s)/4}}{a_r\sim a_s}\,q_{r,s}^2(u),
\end{aligned}
$$

where $a_r\sim a_s,=\binom{A_r}{A_s}(a_r-a_s)$, denotes the difference so chosen that a_s, a_r are in ascending order.

If then we write $M_r=i\sqrt{f'(a_r)/4}$, so that $P'(a_r)=\lambda_r^2 M_r$, we have

$$
\frac{\vartheta^2(u\,|\,u^{a,\,a_r})}{\vartheta^2(u)}=-\frac{1}{M_r}\sum_{i=1}^{p}[c_{p,\,i}+\wp_{p,\,i}(u)]\,a_r^{i-1},
$$

$$
\frac{\vartheta^2(u\,|\,u^{a,\,a_r}+u^{a,\,a_s})}{\vartheta^2(u)}=\frac{\binom{A_r}{A_s}(a_r-a_s)}{M_r M_s}\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{i,\,j}+\wp_{i,\,j}(u)]\,a_r^{i-1}a_s^{j-1};
$$

or, using the sigma functions (Section IX),

$$
\frac{\sigma_r^2(u)}{\sigma^2(u)}=\frac{1}{Q(a_r)}\sum_{i=1}^{p}[c_{p,\,i}+\wp_{p,\,i}(u)]\,a_r^{i-1},
$$

$$
\frac{\sigma_{rs}^2(u)}{\sigma^2(u)}=\frac{1}{Q(a_r)\,Q(a_s)}\sum_{i=1}^{p}\sum_{j=1}^{p}[c_{i,\,j}+\wp_{i,\,j}(u)]\,a_r^{i-1}a_s^{j-1},
$$

where $Q(x)=(x-c_1)\ldots\ldots(x-c_p)(x-c)$.

It is to be noticed that if instead of the function $\vartheta(u)$ we use $\bar{\vartheta}(u)$, $=e^{\frac{1}{2}Cu^2}\vartheta(u)$, where Cu^2 denotes a quadratic form in $u_1,\ldots\ldots,u_p$, $=\sum_{i=1}^{p}\sum_{j=1}^{p}C_{ij}u_i u_j$, then the quadratic form $\frac{1}{2} cu^2$ occurring in the expansion, $\frac{\vartheta(u)}{\vartheta(0)} = 1 + \frac{1}{2} cu^2 + \dots$, is increased by $\frac{1}{2} Cu^2$, and the function $\wp_{ij}(u)$, $= -\partial^2 \log \vartheta(u) / \vartheta u_i \, \partial u_j$ is diminished by C_{ij}. In other words, the quantities $c_{ij} + \wp_{ij}(u)$ are independent of the particular exponential by which we pass from the functions $\Theta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right)$ to the functions $\vartheta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right)$, $= e^{au^2}\Theta\left(u \,\middle|\, \frac{1}{2}\binom{q'}{q}\right)$. This is as it should be, because the quotients $q_r(u)$, $q_{rs}(u)$ are equally independent of the exponential factor e^{au^2}.

The quantities a_{ij} occurring in the exponential e^{au^2}, are such that

$$\int_a^x \int_c^z \frac{dx\,dz}{ys} \, \frac{2ys + F(x, z)}{4(x - z)^2} = \Pi_{z,\,c}^{x,\,a} - 2\sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} \, u_i^{x,\,a} \, u_j^{z,\,c},$$

where $u_1^{x,\,a}, \dots, u_p^{x,\,a}$ are the fundamental integrals of the first kind, $\Pi_{z,\,c}^{x,\,a}$ is Riemann's normal elementary integral of the third kind, and $F(x, z)$ is a rational integral expression, symmetrical in x and z, and of order $p + 1$ in each, which satisfies the equations

$$F(z, z) = 2f(z), \quad \left[\frac{\partial}{\partial x} F(x, z)\right]_{x = z} = \frac{d}{dz} f(z),$$

(cf. B. 194, 315); conversely, any rational integral expression in x and z, of order $p + 1$ in each, which satisfies the equations just written, may be used in the integral

$$\int_a^x \int_c^z \frac{dx\,dz}{ys} \, \frac{2ys + F(x, z)}{4(x - z)^2}$$

and will give rise to corresponding values for the quantities a_{ij}; for the difference of two such expressions as $F(x, z)$ is necessarily of the form

$$(x - z)^2 \, [x, z]_{p-1},$$

where $[x, z]_{p-1}$ denotes a rational integral expression symmetrical in x, z and of order $p - 1$ in each. Particular forms of such an expression $F(x, z)$ are

(α) if

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_{2p+1} x^{2p+1} + \lambda_{2p+2} x^{2p+2},$$

$$F(x, z) = \sum_{i=0}^{p+1} x^i z^i \left[2\lambda_{2i} + \lambda_{2i+1}(x + z)\right];$$

(β) if

$$f(x) = a_x^{2p+2},$$

$$F(x, z) = 2a_x^{p+1} a_z^{p+1};$$

(γ)

$$F(x, z) = 4\,[P(x)\,Q(z) + P(z)\,Q(x)]\,.$$

When the expression $F(x, z)$ is once made precise, or what is the same thing, when the quantities a_{ij} are given, it should be possible to determine the coefficients c_{ij} occurring in the expansion

$$\frac{\vartheta(u)}{\vartheta(0)} = 1 + \tfrac{1}{2}(c_{11}u_1^2 + \ldots. + 2c_{12}u_1u_2 + \ldots.) + \ldots.;$$

we proceed to prove in fact that these coefficients are given by the equation

$$\Sigma\Sigma c_{ij}\,u_i u_j = \frac{4\,[P(\xi_1)\,Q(\xi_2) + P(\xi_2)\,Q(\xi_1)] - F(\xi_1, \xi_2)}{4\,(\xi_1 - \xi_2)^2},$$

where, on the right hand, after the division has been carried out, as is always possible, we are to put

$$\xi_1^{i-1} = u_i, \quad \xi_1^{(0)} = u_1, \quad \xi_2^{i-1} = u_i, \quad \xi_2^{(0)} = u_1.$$

For this we may employ the formula

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\wp_{ij}(u^{x,\,a} + u^{x_1,\,a_1} + \ldots. + u^{x_p,\,a}).\,x_r^{i-1}x_s^{j-1} = \frac{F(x_r, x_s) - 2y_r y_s}{4(x_r - x_s)^2},$$

which is independent of the particular form of the polynomial $F(x, z)$ adopted; this formula is deducible by differentiation from the well-known formula

$$\sum_{i=1}^{p}\Pi_{\bar{z}_i,\,\bar{c}_i}^{x,\,k} = \log\left[\frac{\Theta(u^{x,\,a} + u^{z_1,\,a_1} + \ldots. + u^{z_p,\,a_p})}{\Theta(u^{x,\,a} + u^{c_1,\,a_1} + \ldots. + u^{c_p,\,a_p})}\Big/\frac{\Theta(u^{k,\,a} + u^{z_1,\,a_1} + \ldots + u^{z_p,\,a_p})}{\Theta(u^{k,\,a} + u^{c_1,\,a_1} + \ldots + u^{c_p,\,a_p})}\right],$$

where $\bar{z}_i$, $\bar{c}_i$ denote the places conjugate to z_i and c_i respectively.

It is easy to see, by differentiation of the equation

$$\frac{\vartheta(u)}{\vartheta(0)} = 1 + \tfrac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}c_{ij}\,u_i u_j + \ldots.,$$

that when the arguments are zero,

$$\wp_{ij}(0) = -c_{ij};$$

hence, if in the formula quoted above we put $x = a$, $x_1 = a_1, \ldots, x_p = a_p$, we obtain

$$-\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}\, a_r^{i-1} a_s^{j-1} = + \frac{F(a_r, a_s)}{4(a_r - a_s)^2}, \qquad (\delta)$$

and, similarly, if we put $x = a_s$, $x_1 = a_1, \ldots, x_r = a_r, \ldots, x_s = a, \ldots, x_p = a_p$, we obtain

$$-\sum_{i=1}^{p} c_{p,\,i}\, a_r^{i-1} = \lim_{x=\infty} \frac{F(x, a_r)}{4x^{p-1}(x - a_r)^2}, \qquad (\varepsilon)$$

from the $\frac{1}{2}p(p-1) + p = \frac{1}{2}p(p+1)$ linear equations (δ), (ε), we can determine the $\frac{1}{2}p(p+1)$ quantities c_{ij}; in order then to prove the formula

$$\sum_{i=1}^{p}\sum_{j=1}^{p} c_{ij}\, x^{i-1} z^{j-1} = \frac{4[P(x)\,Q(z) + P(z)\,Q(x)] - F(x, z)}{4(x - z)^2},$$

it is sufficient to prove that it includes the equation (δ), (ε). This is immediately obvious.

In particular when, for

$$f(x) = \lambda + \lambda_1 x + \ldots + \lambda_{2p} x^{2p} + 4x^{2p+1},$$

we take

$$F(x, z) = \sum_{i=0}^{p+1} x^i z^i\, [2\lambda_{2i} + \lambda_{2i+1}(x + z)],$$

we find

$$\sum_{j=1}^{p} c_{p,\,j}\, z^{j-1} = P(z) - z^p,$$

and therefore, if

$$P(x) = x^p + d_1 x^{p-1} + d_2 x^{p-2} + \ldots + d_p,$$

we have

$$c_{p,\,i} = d_{p-i+1}.$$

It follows from these results that the formulæ put down at the commencement of this section can be written

$$-M_r\frac{\vartheta^2(u\,|\,u^{a,\,a_r})}{\vartheta^2(u)}=Q(a_r)\frac{\sigma_r^2(u)}{\sigma^2(u)}=\sum_{i=1}^{p}a_r^{i-1}\wp_{p,\,i}(u)-\lim_{x=\infty}.\frac{F(x,\,a_r)}{4x^{p-1}(x-a_r)^2},$$

$$\frac{M_r\,M_s}{\binom{A_r}{A_s}(a_r-a_s)}\;\frac{\vartheta^2(u\,|\,u^{a,\,a_r}+u^{a,\,a_s})}{\vartheta^2(u)}=Q(a_r)\,Q(a_s)\frac{\sigma_{rs}^2(u)}{\sigma^2(u)}$$

$$=\sum_{i=1}^{p}\sum_{j=1}^{p}a_r^{i-1}a_s^{j-1}\wp_{ij}(u)-\frac{F(a_r,\,a_s)}{4(a_r-a_s)^2};$$

in the second of these the right side is (Section VI) equal to

$$M_r\,M_s\frac{\partial^2}{\partial V_r\,\partial V_s}\log\vartheta(u)-\frac{F(a_r,\,a_s)}{4(a_r-a_s)^2};$$

from the first of these we can deduce the equation satisfied by the values of $x_1,\ldots\ldots,x_p$ which satisfy the inversion equation

$$u_i=u_i^{x_1,\,a}+\ldots\ldots+u_i^{x_p,\,a_p};\qquad\qquad (i=1,\,2,\ldots\ldots,p)$$

for let the coefficient of x^{p+1} in $F(x,\,z)$ be

$$T(z)=t_0z^{p+1}+t_1z^p+t_2z^{p-1}+\ldots\ldots+t_{p+1};$$

then we have

$$\lim_{x=\infty}.\frac{F(x,\,a_r)}{4x^{p-1}(x-a_r)^2}=\tfrac{1}{4}\,T(a_r);$$

further

$$\frac{T(x)}{P(x)}=t_0x+A+\sum_{r=1}^{p}\frac{T(a_r)}{(x-a_r)\,P'(a_r)},$$

where $A=t_1-t_0d_1$,

and therefore

$$\sum_{r=1}^{p}\frac{P(x)\,T(a_r)}{(x-a_r)\,P'(a_r)}=T(x)-(t_0x+A)\,P(x);$$

hence

$$(x-x_1)\ldots\ldots(x-x_p),\;=F(x),$$

$$=P(x)+\sum_{r=1}^{p}\frac{P(x)\,F(a_r)}{(x-a_1)\,P'(a_r)},$$

is (by Section VI) equal to

$$
\begin{aligned}
& P(x) + \sum_{r=1}^{p} \frac{P(x)\, M_r}{(x-a_r)\, P'(a_r)} \; \frac{\vartheta^2(u \,|\, u^{a,\, a_r})}{\vartheta^2(u)}, \\
&= P(x) - \sum_{r=1}^{p} \frac{P(x)}{(x-a_r)\, P'(a_r)} \Big[\sum_{i=1}^{p} a_r^{i-1} \wp_{p,\,i}(u) - \tfrac{1}{4} T(a_r) \Big], \\
&= P(x) + \tfrac{1}{4}\, T(x) - \tfrac{1}{4}(t_0 x + A)\, P(x) - \sum_{i=1}^{p} x^{i-1} \wp_{p,\,i}(u);
\end{aligned}
$$

thus the equation for $x_1, \ldots\ldots, x_p$ is

$$\tfrac{1}{4}\, T(x) - \sum_{i=1}^{p} x^{i-1} \wp_{p,\,i}(u) = \tfrac{1}{4}(t_0 x + A - 4)\, P(x);$$

in particular when

$$F(x, z) = \sum_{i=0}^{p+1} x^i z^i \left[2\lambda_{2i} + \lambda_{2i+1}(x+z) \right],$$

and $\lambda_{2p+2} = 0$, $\lambda_{2p+1} = 4$, we have $T(x) = 4x^p$, $t_0 = 0$, $t_1 = 4$, $A = 4$, and the equation becomes

$$x^p - x^{p-1} \wp_{p,\,p}(u) - x^{p-2} \wp_{p,\,p-1}(u) - \ldots\ldots - \wp_{p,\,1}(u) = 0.$$

(Cf. Bolza, American Journal, XVII (1895), and B. 324.)

For the sake of completeness we put here also a particular case of a formula given by Bolza (Göttingen, Nachrichten, 1894); from the $\frac{1}{2} p(p+1)$ equations expressed by

$$
\begin{aligned}
\sum_{i=1}^{p} \sum_{j=1}^{p} \wp_{ij}(u)\, x_r^{i-1} x_s^{j-1} &= \frac{F(x_r, x_s) - 2y_r y_s}{4(x_r - x_s)^2}, \\
\sum_{i=1}^{p} \wp_{p,\,i}(u)\, x_r^{i-1} &= \lim_{x=\infty} \Big[\frac{F(x, x_r)}{4x^{p-1}(x - x_r)^2} \Big],
\end{aligned}
$$

where $$u_i = u_i^{x_1,\, a_1} + \ldots\ldots + u_i^{x_p,\, a_p}, \qquad (i = 1, 2, \ldots\ldots, p)$$

and x_r, x_s are chosen from $x_1, \ldots\ldots, x_p$, we can obtain the algebraic expressions of the $\frac{1}{2}p(p+1)$ quantities $\wp_{ij}(u)$; the result is that, *for any values of x and z*, we have, with $F(x) = (x - x_1) \dots (x - x_p)$,

$$\sum_{i=1}^{p} \sum_{j=1}^{p} [\wp_{ij}(u) + c_{ij}]\, x^{i-1} z^{j-1} = \tfrac{1}{4} F(x)\, F(z) \Big[\sum_{i=1}^{p} \frac{y_i}{(x_i - x)(x_i - z)\, F'(x_i)}\Big]^2$$

$$- \frac{[F(x)\, P(z) - F(z)\, P(x)][F(x)\, Q(z) - F(z)\, Q(x)]}{(x-z)^2 F(x)\, F(z)};$$

to prove this it is sufficient to see that this last result includes the $\frac{1}{2}p(p+1)$ equations before given. Now putting x_r, x_s for x, z, the right side of the formula last written becomes

$$\tfrac{1}{4} F(x)\, F(z) \Big[\frac{y_r}{(x_r - x)(x_r - z)\, F'(x_r)} + \frac{y_s}{(x_s - x)(x_s - z)\, F'(x_s)}\Big]^2,$$

$$-\tfrac{1}{4} \frac{f(z)}{(x-z)^2} \frac{F(x)}{F(z)} - \tfrac{1}{4} \frac{f(x)}{(x-z)^2} \frac{F(z)}{F(x)} + \frac{P(x)\, Q(z) + P(z)\, Q(x)}{(x-z)^2},$$

$$= \frac{4[P(x_r)\, Q(x_s) + P(x_s)\, Q(x_r)] - 2y_r y_s}{4(x_r - x_s)^2},$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij}\, x_r^{i-1} x_s^{j-1} + \frac{F(x_r, x_s) - 2y_r y_s}{4(x_r - x_s)^2},$$

which is in accordance with $\frac{1}{2}p(p-1)$ of the equations in question; while, dividing by x^{p-1} and putting $z = x_r$, we obtain

$$\sum_{i=1}^{p} [\wp_{p,i}(u) + c_{p,i}]\, x_r^{i-1} = P(x_r) = \lim_{x=\infty} \frac{P(x)\, Q(x_r) + P(x_r)\, Q(x)}{(x - x_r)^2},$$

$$= \sum_{i=1}^{p} c_{p,i}\, x_r^{i-1} + \lim_{x=\infty} \frac{F(x, x_r)}{4x^{p-1}(x - x_r)^2},$$

which is in accordance with the last p equations in question.

In conclusion we calculate the values of c_{11}, c_{12}, c_{22} for $p = 2$, with various hypotheses in regard to $F(x, z)$.

When

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + 4x^5$$

$$\begin{aligned} F(x, z) &= \sum_{=0}^{p+1} x^i z^i [2\lambda_{2i} + \lambda_{2i+1}(x+z)] \\ &= 2\lambda + \lambda_1(x+z) + xz[2\lambda_2 + \lambda_3(x+z)] + x^2 z^2 [2\lambda_4 + 4(x+z)], \end{aligned}$$

we have already found

$$c_{2,1} = c_{1,2} = a_1 a_2, \quad c_{2,2} = -a_1 - a_2;$$

thence from

$$c_{11} + c_{12}(a_1 + a_2) + c_{22} a_1 a_2 = -\frac{F(a_1, a_2)}{4(a_1 - a_2)^2},$$

we have

$$c_{11} = -\frac{F(a_1, a_2)}{4(a_1 - a_2)^2};$$

now

$$\begin{aligned} &F(x, z) - f(x) - f(z) \\ &\quad = (x-z)^2 [-\lambda_2 - \lambda_3(x+z) - \lambda_4(x+z)^2 - 4(x+z)(x^2 + xz + z^2)], \end{aligned}$$

and therefore

$$4c_{11} = \lambda_2 + \lambda_3(a_1 + a_2) + \lambda_4(a_1 + a_2)^2 + 4(a_1 + a_2)(a_1^2 + a_1 a_2 + a_2^2),$$

or, if we write

$$f(x) = 4(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5),$$

then

$$\begin{aligned} c_{11} = 10A_3 + 10A_2(a_1 + a_2) &+ 5A_1(a_1 + a_2)^2 \\ &+ (a_1 + a_2)(a_1^2 + a_1 a_2 + a_2^2). \end{aligned}$$

If, however, with

$$f(x) = a_x^6 = 4p_x^3 q_x^3,$$

we take

$$F(x, z) = 2a_x^3 a_z^3,$$

then

$$\begin{aligned} &c_{11} + c_{12}(x+z) + c_{22} xz, \\ &= \frac{p_x^3 q_z^3 + p_z^3 q_x^3 - \frac{1}{2} a_x^3 a_z^3}{(x-z)^2}, \\ &= \tfrac{9}{10}(pq)^2 (p_x q_z + p_z q_x), \end{aligned}$$

as can easily be calculated. Hence, putting

$$f(x) = 4(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5),$$

we find, with $h_1 = -a_1 - a_2$, $h_2 = a_1 a_2$, that

$$c_{22} = h_1 - 2A_1, \quad c_{12} = h_2 - A_2, \quad c_{11} = -h_1^3 + 5h_1^2 A_1 - 10h_1 A_2 + h_1 h_2 + 6A_3;$$

if we denote these by $\overline{c_{22}}$, $\overline{c_{12}}$, $\overline{c_{11}}$, and those just found by c_{22}, c_{12}, c_{11}, we have

$$c_{22} - \overline{c}_{22} = 2A_1, \quad c_{12} - \overline{c}_{12} = A_2, \quad c_{11} - \overline{c}_{11} = 4A_3.$$

In other words, the theta functions formed with $F(x, z)$ equal to the former value are obtained from those formed with $F(x, z) = 2a_x^3 a_z^3$, by multiplying by the factor

$$e^{A_1 u_2^2 + A_2 u_1 u_2 + 2A_3 u_1^2}.$$

SECTION XIV.

Evaluation of $\vartheta(u+v)\,\vartheta(u-v)/\vartheta^2(u)\,\vartheta^2(v)$ in certain cases.

Putting $\vartheta_{12}(u) = \vartheta(u \,|\, u^{a,\,a_1} + u^{a,\,a_2})$ we work out first the case, in which $p = 2$, of

$$\frac{\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)}.$$

From the addition formula (Section XI), by adding the half-period $u^{a,\,a_1} + u^{a,\,a_2}$ to the argument u, and dividing by $\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)$ we obtain, on the hypothesis of the dissection (I),

$$\frac{\vartheta^2(0)\,\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)} = \frac{\theta^2}{\theta_{12}^2} - \frac{\vartheta^2}{\vartheta_{12}^2} - \frac{\vartheta_2^2}{\vartheta_{12}^2} \cdot \frac{\theta_1^2}{\theta_{12}^2} + \frac{\vartheta_1^2}{\vartheta_{12}^2} \cdot \frac{\theta_2^2}{\theta_{12}^2},$$

where ϑ, ϑ_1, ... denote $\vartheta(u)$, $\vartheta(u \,|\, u^{a,\,a_1})$, ... and θ, θ_1, ... denote $\vartheta(v)$, $\vartheta(v \,|\, u^{a,\,a_1})$, ... Now if L_{ij} be defined by

$$\begin{aligned} L_{ij} &= -\frac{\partial^2}{\partial u_i\,\partial u_j} \log \vartheta_{12}(u) + c_{ij}, \\ &= \wp_{ij}(u \,|\, u^{a,\,a_1} + u^{a,\,a_2}) + c_{ij}, \end{aligned}$$

we have (Section XIII)

$$-\frac{\vartheta_2^2}{\vartheta_{12}^2} = -\frac{1}{M_1}(L_{22} a_1 + L_{21}), \quad \frac{\vartheta_1^2}{\vartheta_{12}^2} = -\frac{1}{M_2}(L_{22} a_2 + L_{21}),$$

$$-\frac{\vartheta^2}{\vartheta_{12}^2} = \frac{a_2 - a_1}{M_1 M_2}\left[L_{22} a_1 a_2 + L_{21}(a_1 + a_2) + L_{11}\right];$$

hence

$$\frac{M_1 M_2}{a_2 - a_1} \frac{\vartheta^2(0)\,\vartheta_{12}(u+v)\,\vartheta_{12}(u-v)}{\vartheta_{12}^2(u)\,\vartheta_{12}^2(v)}$$
$$= (L_{12} - L'_{22})\,a_1 a_2 + (L_{21} - L'_{21})(a_1 + a_2) + L_{11} - L'_{11}$$
$$+ \frac{1}{a_2 - a_1}\left[(L_{22}\,a_1 + L_{21})(L'_{22}\,a_2 + L'_{21}) - (L'_{22}\,a_1 + L'_{21})(L_{22}\,a_2 + L_{21})\right],$$

where L'_{ij} is the same function of v that L_{ij} is of u; and if we use $\wp_{ij}$ to denote $\wp_{ij}(u \,|\, u^{a,\,a_1} + u^{a,\,a_2})$, and $\wp'_{ij}$ to denote $\wp_{ij}(v \,|\, u^{a,\,a_1} + u^{a,\,a_2})$, this gives

$$(\wp_{22} - \wp'_{22})\,a_1 a_2 + (\wp_{21} - \wp'_{21})(a_1 + a_2) + \wp_{11} - \wp'_{11}$$
$$+ (\wp_{21} + c_{21})(\wp'_{22} + c_{22}) - (\wp'_{21} + c_{21})(\wp_{22} + c_{22}),$$
$$= \wp_{11} - \wp'_{11} + (\wp_{21} - \wp'_{21})(a_1 + a_2 + c_{22}) + (\wp_{22} - \wp'_{22})(a_1 a_2 - c_{21}) + \wp_{21}\wp'_{22} - \wp_{22}\wp'_{21}.$$

Recalling the definition of the σ-function (Section IX), we therefore have the result

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + [\wp_{21}(u) - \wp_{21}(v)](a_1 + a_2 + c_{22})$$
$$+ [\wp_{22}(u) - \wp_{22}(v)][a_1 a_2 - c_{21}]$$
$$+ \wp_{21}(u)\,\wp_{22}(v) - \wp_{22}(u)\,\wp_{21}(v).$$

Hence, if

$$f(x) = \lambda + \lambda_1 x + \lambda_2 x^2 + \ldots. + \lambda_4 x^4 + 4x^5,$$
$$F(x, z) = \sum_{i=0}^{p} x^i z^i\,[2\lambda_{2i} + \lambda_{2i+1}(x + z),$$

we have (Section XIII)

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) + \wp_{21}(u)\,\wp_{22}(v) - \wp_{22}(u)\,\wp_{21}(v);$$

while if

$$f(x) = 4\,(x^5 + 5A_1 x^4 + 10A_2 x^3 + 10A_3 x^2 + 5A_4 x + A_5)$$
$$= a_x^6,$$

and $F(x, z) = 2a_x^3 a_z^3$,

we similarly have

$$-\frac{\sigma_{12}(u+v)\,\sigma_{12}(u-v)}{\sigma_{12}^2(u)\,\sigma_{12}^2(v)} = \wp_{11}(u) - \wp_{11}(v) - 2A_1\,[\wp_{12}(u) - \wp_{12}(v)]$$
$$+ A_2\,[\wp_{22}(u) - \wp_{22}(v)] + \wp_{21}(u)\,\wp_{22}(v) - \wp_{21}(v)\,\wp_{22}(u).$$

In either of these the function $\wp_{ij}(u)$ is given by

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \, \partial u_j} \log \sigma_{12}(u).$$

The forms here obtained for the right sides in these equations can be variously modified by utilizing the fact that the three functions $\wp_{11}(u)$, $\wp_{12}(u)$, $\wp_{22}(u)$ are connected by an algebraic equation.

Consider next the case, for $p = 3$, of the function

$$\frac{\vartheta_{123}(u+v)\,\vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\,\vartheta_{123}^2(v)},$$

where $\vartheta_{123}(u) = \vartheta(u \,|\, u^{a,\, a_1} + u^{a,\, a_2} + u^{a,\, a_3})$. We put, in the course of the work, ϑ_{123} for $\vartheta_{123}(u)$ and θ_{123} for $\vartheta_{123}(v)$, etc.; further, we put

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \, \partial u_j} \log \vartheta_{123}(u) = \wp_{ij},$$

$$\wp_{ij}(v) = -\frac{\partial^2}{\partial v_i \, \partial v_j} \log \vartheta_{123}(v) = \wp'_{ij},$$

and

$$\begin{array}{ll} A = \wp_{11} - \wp'_{11}, & F = \wp_{23} - \wp'_{23}, \\ B = \wp_{22} - \wp'_{22}, & G = \wp_{31} - \wp'_{31}, \\ C = \wp_{33} - \wp'_{33}, & H = \wp_{12} - \wp'_{12}. \end{array}$$

We have, for the dissection (I),

$$\vartheta^2 \vartheta_{123}^2 = \vartheta_1^2 \vartheta_{23}^2 - \vartheta_2^2 \vartheta_{31}^2 + \vartheta_3^2 \vartheta_{12}^2,$$

and therefore

$$\frac{\vartheta^2}{\vartheta_{123}^2} = \frac{\vartheta_1^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{23}^2}{\vartheta_{123}^2} - \frac{\vartheta_2^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{31}^2}{\vartheta_{123}^2} + \frac{\vartheta_3^2}{\vartheta_{123}^2} \cdot \frac{\vartheta_{12}^2}{\vartheta_{123}^2};$$

also, by the addition formula (Section XI), by increasing the argument u by the half-period $u^{a,\, a_1} + u^{a,\, a_2} + u^{a,\, a_3}$, we have

$$\frac{\vartheta^2(0)\,\vartheta_{123}(u+v)\,\vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\,\vartheta_{123}^2(v)}$$

$$= \frac{\theta^2}{\theta_{123}^2} + \frac{\vartheta^2}{\vartheta_{123}^2} - \frac{\vartheta_{23}^2}{\vartheta_{123}^2}\,\frac{\theta_1^2}{\theta_{123}^2} - \frac{\vartheta_1^2}{\vartheta_{123}^2}\,\frac{\theta_{23}^2}{\theta_{123}^2} + \frac{\vartheta_{31}^2}{\vartheta_{123}^2}\,\frac{\theta_2^2}{\theta_{123}^2}$$

$$+ \frac{\vartheta_2^2}{\vartheta_{123}^2}\,\frac{\theta_{31}^2}{\theta_{123}^2} - \frac{\vartheta_{12}^2}{\vartheta_{123}^2}\,\frac{\theta_3^2}{\theta_{123}^2} - \frac{\vartheta_3^2}{\vartheta_{123}^2}\,\frac{\theta_{12}^2}{\theta_{123}^2}$$

$$= \left(\frac{\vartheta_1^2}{\vartheta_{123}^2} - \frac{\theta_1^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{23}^2}{\vartheta_{123}^2} - \frac{\theta_{23}^2}{\theta_{123}^2}\right) - \left(\frac{\vartheta_2^2}{\vartheta_{123}^2} - \frac{\theta_2^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{13}^2}{\vartheta_{123}^2} - \frac{\theta_{13}^2}{\theta_{123}^2}\right)$$

$$+ \left(\frac{\vartheta_3^2}{\vartheta_{123}^2} - \frac{\theta_3^2}{\theta_{123}^2}\right)\left(\frac{\vartheta_{12}^2}{\vartheta_{123}^2} - \frac{\theta_{12}^2}{\theta_{123}^2}\right);$$

now we have proved (Section XIII) that

$$\frac{\vartheta_{rs}^2(u)}{\vartheta^2(u)} = \frac{\binom{A_r}{A_s}(a_r - a_s)}{M_r M_s} \sum_{i=1}^{p} \sum_{j=1}^{p} a_r^{i-1} a_s^{j-1} \left[c_{ij} - \frac{\partial^2}{\partial u_i \partial u_j} \log \vartheta(u)\right],$$

$$\frac{\vartheta_2^2(u)}{\vartheta_r^2(u)} = -\frac{1}{M_r} \sum_{i=1}^{p} a_r^{i-1} \left[c_{p,\,i} - \frac{\partial^2}{\partial u_p \partial u_i} \log \vartheta(u)\right];$$

hence if r, s, t denote 1, 2, 3, in some order, we have, the function $\wp_{ij}(u)$ being, as before stated, derived from $\vartheta_{123}(u)$,

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ & A_r & A_s \end{pmatrix} \frac{\vartheta_t^2(u)}{\vartheta_{123}^2(u)} = \frac{\binom{A_r}{A_s}(a_r - a_s)}{M_r M_s} \sum_{i=1}^{p} \sum_{j=1}^{p} a_r^{i-1} a_s^{j-1} \left[c_{ij} + \wp_{ij}(u)\right],$$

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ & & A_t \end{pmatrix} \frac{\vartheta_{rs}^2(u)}{\vartheta_{123}^2(u)} = -\frac{1}{M_t} \sum_{i=1}^{p} a_t^{i-1} \left[c_{p,\,i} + \wp_{p,\,i}(u)\right];$$

thus, in the notation previously explained,

$$\begin{aligned} &M_1 M_2 M_3 \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)} \\ &= \Sigma (a_2 - a_3)[a_1^2 C + a_1 F + G][A + B a_2 a_3 + C a_2^2 a_3^2 + F a_2 a_3 (a_2 + a_3) \\ &\qquad\qquad + G(a_2^2 + a_3^2) + H(a_2 + a_3)], \end{aligned}$$

where the sign of summation refers to the three suffixes a_1, a_2, a_3, to be taken in cyclical order,

$$= (a_2 - a_3)(a_3 - a_1)(a_1 - a_2)[HF - BG - CA + G^2],$$

as can be immediately verified; now we have seen (Section VII) that

$$\sigma_{123}(u) = \frac{\lambda_1 \lambda_2 \lambda_3}{\zeta_{23} \zeta_{31} \zeta_{12}} \frac{\vartheta_{123}(u)}{\vartheta(0)},$$

where $\lambda_r^2 M_r = P'(a_r) = (a_r - a_s)(a_r - a_t)$;

hence

$$\begin{aligned} \frac{\sigma_{123}(u+v)\, \sigma_{123}(u-v)}{\sigma_{123}^2(u)\, \sigma_{123}^2(v)} &= \frac{(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)}, \\ &= -\frac{M_1 M_2 M_3}{(a_2 - a_3)(a_3 - a_1)(a_1 - a_2)} \frac{\vartheta^2(0)\, \vartheta_{123}(u+v)\, \vartheta_{123}(u-v)}{\vartheta_{123}^2(u)\, \vartheta_{123}^2(v)}; \end{aligned}$$

thus, finally,

$$
\begin{aligned}
-\frac{\sigma_{123}(u+v)\,\sigma_{123}(u-v)}{\sigma_{123}^2(u)\,\sigma_{123}^2(v)} = {} & [\wp_{31}(u)-\wp_{31}(v)]^2 - [\wp_{31}(u)-\wp_{31}(v)][\wp_{22}(u)-\wp_{22}(v)] \\
& - [\wp_{33}(u)-\wp_{33}(v)][\wp_{11}(u)-\wp_{11}(v)] \\
& + [\wp_{12}(u)-\wp_{12}(v)][\wp_{23}(u)-\wp_{23}(v)]\,.
\end{aligned}
$$

Other cases can be similarly worked out. In accordance however with the results enunciated by Weierstrass, Crelle LXXXIX (1880), every $p+1$ of the functions $\wp_{ij}(u)$ are connected by an algebraical equation, which is in fact not difficult to obtain. It becomes therefore proper to investigate the modifications following from the introduction of these algebraical equations. For the present paper it is sufficient to have investigated a method whereby the actual equations can always be calculated.

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